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Supersymmetric states of $\mathcal{N} = 4$ Yang-Mills from giant gravitons

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Abstract: Mikhailov has constructed an infinite family of $\frac{1}{8}$ BPS D3-branes in $AdS_5 \times S^5$. We regulate Mikhailov's solution space by focussing on finite dimensional submanifolds. Our submanifolds are topologically complex projective spaces with symplectic form cohomologically equal to $2\pi N$ times the Fubini-Study Kähler class. Upon quantization and removing the regulator we find the Hilbert Space of N noninteracting Bose particles in a 3d Harmonic oscillator, a result previously conjectured by Beasley. This Hilbert Space is isomorphic to the classical chiral ring of $\frac{1}{8}$ BPS states in $\mathcal{N} = 4$ Yang-Mills theory. We view our result as evidence that the spectrum of $\frac{1}{8}$ BPS states in $\mathcal{N} = 4$ Yang Mills theory, which is known to jump discontinuously from zero to infinitesimal coupling, receives no further renormalization at finite values of the 't Hooft coupling.

Keywords: AdS-CFT Correspondence, D-branes, Supersymmetry and Duality, Gauge-gravity correspondence.

Contents

1. Introduction and summary

The $\mathrm{U}(N)$, $\mathcal{N} = 4$ Yang-Mills theories are special in many ways. These theories occur in fixed lines and are maximally supersymmetric (enjoying invariance under the 32 supercharge supergroup PSU(2,2|4)) for every value of the coupling constant [1 – 4]. They exhibit invariance under S duality [5 – 8]. In the large N 't Hooft limit they also admit a dual reformulation as a weakly coupled string theory, a description which in turn is well approximated by supergravity on $AdS_5 \times S^5$ at large values of the 't Hooft coupling $\lambda = g^2 N$ [9 – 12]. Finally the appearance of integrable structures in recent studies of perturbation theory in the 't Hooft limit suggests that these theories may be unusually tractable, and may even be 'solvable' in appropriate limits (see, for instance, [13 – 18]). In summary $\mathcal{N} = 4$ Yang-Mills are both intensely interesting as well as unusually tractable and deserve to be thoroughly studied.

The $\mathcal{N} = 4$ Yang-Mills theories on S^3 possess an infinite number of special states, distinguished by the fact that they are annihilated by some fraction of the supersymmetry generators (together with the Hermitian conjugates of these supercharges). As the $\mathcal{N} = 4$ theory is special largely because of its high degree of supersymmetry, the study of its supersymmetric states may well prove particularly tractable and rewarding.

The simplest supersymmetric states are annihilated by half of the 16 supercharges together with their Hermitian conjugates [19]. In the absence of a phase transition, it follows immediately from PSU(2,2|4) representation theory that the spectrum of half BPS states cannot change under continuous variations of a parameter. As a consequence the spectrum of half BPS spectrum of $\mathcal{N} = 4$ Yang-Mills, at all values of the coupling, may be enumerated by via a simple counting in the free theory.

Less special supersymmetric states may be annihilated by a quarter [20 – 22], an eighth or a sixteenth of the supercharges and their Hermitian conjugates. The spectrum of these states can, in general, vary as a function of the coupling. Indeed the spectrum of one fourth, one eight and one sixteenth BPS states in $\mathrm{U}(N)$, $\mathcal{N} = 4$ Yang-Mills at infinitesimal coupling is discontinuously smaller than the same spectrum in the free theory. It was suggested by the authors of [23], that this spectrum receives no further renormalization at finite coupling. According to this conjecture $\mathrm{U}(N)$, $\mathcal{N} = 4$ Yang-Mills theories have the same spectrum of supersymmetric states at every finite, nonzero value of the Yang-Mills coupling.

The authors of [23] also computed an exact (finite N) formula for the finite coupling partition function over the classical one eighth BPS chiral ring of $\mathrm{U}(N)$ Yang-Mills theory. The formula in [23] turns out to reproduce the spectrum of one eighth BPS multi-gravitons in $AdS_5 \times S^5$ (a description of the theory that is accurate at large λ) at energies much smaller than N, but disagrees with this spectrum at energies of order N or higher. Nonetheless the authors of [23] proposed that their classical chiral ring partition function was equal, at all finite nonzero values of the 't Hooft coupling, to the partition function over one eighth BPS states in $\mathrm{U}(N)$, $\mathcal{N} = 4$ Yang Mills theory. This proposal is not in conflict with the AdS/CFT correspondence. Gravitons propagating with energies of order N in $AdS_5 \times S^5$ blow up into puffed up $D3$ branes; so called giant gravitons [24 – 27]. As a consequence supersymmetric spectrum of IIB theory on $AdS_5 \times S^5$ at energies of order N or higher,

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cannot be determined by the Kaluza-Klein reduction of IIB theory on $AdS_5 \times S^5$, but instead requires a detailed study of the moduli space of giant gravitons; the content of this paper.

In a beautiful paper written over five years ago, Mikhailov [28] constructed a large class of classical solutions for $\frac{1}{8}$ BPS giant gravitons in $AdS_5 \times S^5$, which we will now briefly review. Consider (generically 3 dimensional) surfaces defined by the intersection of the zero set (i.e. pre-image of zero) of any holomorphic function in $\mathbb{C}^3$ with the five dimensional unit sphere $\sum_{i=1}^{3} |z^i|^2 = 1$. Mikhailov demonstrated that D3-branes that wrap any such surface on the S^5 of $AdS_5 \times S^5$, with the world volume field strength set to zero, are supersymmetric. Four years ago, Beasley [29] already made an insightful conjecture about the quantization of these solutions. In this paper we perform a detailed study of the quantization of moduli space of holomorphic surface in $\mathbb{C}^3$ that intersect the unit sphere, with respect to the symplectic form determined by the world volume action of the D3-brane; our results are in complete agreement with Beasley's conjecture. In particular we demonstrate that the Hilbert space so obtained exactly reproduces the (appropriate restriction of) the partition function over the classical chiral ring in [23]. In our opinion, this result provides significant evidence in support of the finite N, finite λ partition function over one eighth BPS states proposed in [23].

We highlight physically interesting aspects of our results in the rest of this subsection, postponing a summary of our technical results to the next subsection. As we have described above, we have quantized a manifold of supersymmetric solutions. Now the quantization of an arbitrary submanifold of solution space certainly does not, in general, produce any subspace of the full Hilbert space. However there is circumstantial evidence (see, for instance, [30 − 32]) that supersymmetric solutions are special; that the the Hilbert Space obtained from the quantization of the set of all supersymmetric solutions of a system is equal to the restriction to supersymmetric states of the full Hilbert Space. The results of this paper may be taken as further evidence for this statement. It would be useful and interesting to have better understanding of this issue.

Mikhailov's solutions do not exhaust the full set of one eighth BPS configurations of D3-branes for two separate reasons. First there exists a disconnected branch of one eighth BPS giant D3-brane configurations, the so called dual giant gravitons. As we will outline in section 8, as yet unpublished work of G. Mandal and N. Suryanarayana [33] strongly suggests that the partition function of [23] may also be obtained by quantizing the manifold of one eighth BPS dual giant gravitons! Thus the familiar (but still slightly mysterious) duality between half BPS giants and dual giants appears to extend to one eighth BPS states. It would be certainly be interesting to understand this better.

The second incompleteness in Mikhailov's space of supersymmetric solutions has to do with one eighth BPS D3-brane configurations which involve world volume fermions and gauge fields. Mikhailov's construction describes only those $\frac{1}{8}$ BPS D-brane configurations with worldvolume fermions and gauge fields set to zero and so the quantization of these solutions produces only the part of the $\frac{1}{8}$ BPS cohomology constructed from the three chiral Yang-Mills scalar fields. In order to recover all other states in the $\frac{1}{8}$ BPS cohomology, one could complete the construction of all supersymmetric classical solutions with worldvolume fermions and gauge fields turned on (see [34] for work in this direction) and quantize this manifold of solutions. Although we have not carried out this procedure, for our purposes it seems almost redundant; as we will now describe, supergroup representation theory almost guarantees that this procedure will reproduce all the remaining states in the cohomology of [23].

The full set of one eighth BPS states appear in representations of SU(2|3) subgroup that commutes with the supercharges that annihilate these states. In appendix B we demonstrate that the full $\frac{1}{8}$ BPS cohomology may be obtained by acting on states built out of the chiral scalars with the generators of the SU(2|3) subgroup of PSU(2,2|4) of supercharges that commute with those that are annihilated on $\frac{1}{8}$ cohomology. As a consequence, any quantization procedure that respect SU(2|3) invariance is almost guaranteed to fill out the remaining states in the cohomology listed in [23].[1]

We have quantized the supersymmetric motion of the $D3$-brane on the S^5 of $AdS_5 \times S^5$. We would like to emphasize that the restriction to the supersymmetric sector retains several of the complications that have plagued previous attempts at quantizing $p \geq 2$ branes. In particular smooth variations of the parameters of the holomorphic polynomial can cause the surface of the $D3$ brane to undergo topology changing transitions. Our quantization procedure manages to deal with these transitions in a smooth way ([35] made similar remarks in another context). It is possible that a detailed study of the quantization in the supersymmetric sector will throw up lessons of relevance to quantization of higher dimensional surfaces in general.

In this paper we have demonstrated that the quantization of giant gravitons yields the same BPS spectrum as weakly coupled Yang-Mills. It would be fascinating if we could see Mikhailov's holomorphic surfaces emerge more directly from an analysis of the gauge theory (see [36 − 39] for related work).

Finally, it would be natural, and extremely interesting, to attempt to extend our work to the quantization of $\frac{1}{16}$ supersymmetric giant gravitons. The potential payoffs of such an extension are large, as $\frac{1}{16}$ BPS states in $\mathcal{N} = 4$ Yang-Mills are much richer and much less well understood than their $\frac{1}{8}$ BPS counterparts. In particular there exist smooth $\frac{1}{16}$ BPS black holes in $AdS_5 \times S^5$ (see [40 − 44] and references therein), whose entropy has not yet successfully been accounted for (see [23, 45] for a recent discussion).

1.1 Technical aspects

In this subsection we will briefly summarize our technical constructions and results.

We first describe how the D3-brane surfaces described by Mikhailov may be algebraically parameterized. By the Weierstrass approximation theorem any surface $f(z^i) = 0$ that is holomorphic in an open shell surrounding the unit sphere in $\mathbb{C}^3$ may be approximated to arbitrary accuracy (in that neighbourhood) by a sequence of surfaces $P_m(z^i) = 0$ where $P_m(z^i)$ are polynomials in variables z^i $(i = 1, 2, 3)$. As a consequence the set of all supersymmetric configurations of D3-branes is generated by the intersections of arbitrary

[1]This argument makes the reasonable assumption that the states obtained from the quantization of supersymmetric D3-branes with world volume fermions and gauge fields turned off are all SU(2|3) primaries, and, moreover, are the only such primaries. A direct demonstration of this point would allow us to remove the word 'almost' in this paragraph.

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polynomial surfaces $P(z^i) = 0$ with the unit sphere. We will find it useful, in this paper, to regulate this set of surfaces by studying the linear set of polynomials, P_C, generated by arbitrary linear combinations of n_C arbitrarily specified monomials (we denote the set of 3 tuples (n_1, n_2, n_3) by C, so the set of monomials is $\{(z^1)^{n_1}(z^2)^{n_2}(z^3)^{n_3} \mid \vec{n} \in C\}$ and P_C is its linear span) [2]. In this paper we determine the Hilbert Space $\mathcal{H}_C$ obtained by quantizing $P(z) = 0$ with the unit sphere for $P \in P_C$.

We will now describe the set of intersections of $P(z^i) = 0$ (for $P(z)$ in P_C) with the unit sphere, and the associated Hilbert Space $\mathcal{H}_C$ in more detail. Note that the zero sets of the polynomials in P_C are left unchanged by an overall rescaling of the coefficients in the polynomial; as a consequence these zero sets are in one to one correspondence with $\mathbb{CP}^{n_C-1}$, as already noted by Beasley [29]. However, not all of these complex surfaces $\{(z_1, z_2, z_3) \mid P(z_1, z_2, z_3) = 0\}$, $P \in P_C$, intersect the unit five sphere S^5. Polynomials $P(z)$ that give surfaces that do not intersect with S^5 do not belong to the space to be quantized; as a consequence the space of interest is the projective space of polynomial coefficients with holes eaten out. Furthermore even those surfaces, $P(z^i) = 0$, that do intersect the unit 5-sphere are not parameterized in a one to one fashion by polynomials $P(z^i)$; there exist degenerate families of polynomials, all of whose surfaces have the same (nonzero) intersection with the unit S^5 [3]. In addition, the symplectic form also has singularities along certain surfaces (e.g. where the topology of the intersection changes). Despite all these apparent complications, we show in sections 2 and 3 below that the space of intersections is extremely simple. In particular we demonstrate that the space in question is topologically $\mathbb{CP}^{n_C-1}$ and that, in an appropriate sense, the symplectic form is well defined, $\mathrm{U}(3)$ invariant, everywhere invertible, and in the cohomology class of $(2\pi N)\omega_{\mathrm{FS}}$, where ω_{FS} is the usual[4] Fubini-Study form on $\mathbb{CP}^{n_C-1}$. In sections 5, 6 and 7 we illustrate and illuminate aspects of the mathematically abstract arguments of sections 2 and 3 by an independent direct and detailed study of three specially chosen subfamilies of polynomials P_C (and their quantization, see below).

It follows almost immediately (see section 4.1) that $\mathcal{H}_C$ is isomorphic to the Hilbert space obtained from holomorphic quantization of $\mathbb{CP}^{n_C-1}$ equipped with the symplectic form $(2\pi N)\omega_{\mathrm{FS}}$. It is well known that the latter may be identified with the set of homogeneous polynomials, of degree N of the n_C projective coordinates $\{w_{\vec{n}} \mid \vec{n} \in C\}$ of $\mathbb{CP}^{n_C-1}$ ($w_{\vec{n}}$ is related to the coefficient of $(z^1)^{n_1}(z^2)^{n_2}(z^3)^{n_3}$ in the set of polynomials and, in particular, has the same $\mathrm{U}(3)$ transformation properties). After quantization, the charge operators are $\sum_{\vec{n}\in C} n_m w^{\vec{n}} \partial_{w_{\vec{n}}}$, where n_m is the charge of $w_{\vec{n}}$ under the $\mathrm{U}(1)$ rotation in the m^{th} independent two-plane ($m = 1 \ldots 3$) in $\mathbb{R}^6 = \mathbb{C}^3$. Consequently the charge of a degree N monomial in $w_{\vec{n}}$'s is simply the sum over the charges n_m, over the N factor $w_{\vec{n}}$'s of the monomial. Thus $\mathcal{H}_C$ is isomorphic to the Hilbert space of N identical noninteracting bosons with an n_C dimensional single particle Hilbert space whose states have charges $\vec{n}$.

[2]For example P_{C_k} could be the set of all polynomials in three variables of degree at most k; in this case $n_C = \binom{k+3}{3} = (k+3)(k+2)(k+1)/3!$).

[3]The 'holes' referred to above are a special case; polynomials that do not intersect the unit S^5 all have equal (empty) intersections with it.

[4]Note that the Fubini-Study metric defined in [46] is a factor of 2π larger than that in, e.g. [47]. We use the latter normalization.

Note that the regulated Hilbert Spaces $\mathcal{H}_C$ have the following inclusivity property. If two 3 tuple sets obey $C_{k_1} \subset C_{k_2}$ then $\mathcal{H}_{k_1} \subset \mathcal{H}_{k_2}$. Now consider a sequence of 3 tuple sets C_{k_i} chosen so that $C_{k_1} \subset C_{k_2}$ for $k_1 < k_2$ and with the property that any particular polynomial $P(z)$ is contained in P_{C_k} for some large enough k. The completion of the direct limit, namely $\mathcal{H} = \lim_{k\to\infty} \mathcal{H}_k$, is a unique, well defined (indeed familiar) infinite dimensional Hilbert space, which may thus be regarded as the Hilbert space of Mikhailov's giant gravitons.

The maximal Hilbert Space $\mathcal{H}$ is especially familiar. In this case the single particle Hilbert Space is simply the Hilbert space of a three dimensional harmonic oscillator, with the 3 U(1) charges identified with the excitation number operators of the three oscillators. As a consequence, the quantization of Mikhailov's solutions yields the Hilbert space of N identical bosons in a 3d harmonic oscillator, and is identical to that part of $\frac{1}{8}$ BPS Hilbert space of $\mathcal{N} = 4$ Yang Mills in the conjectured formula of [23], that is made up entirely out of the three holomorphic scalar fields (see [29] for related remarks).

2. The symplectic form and charges

2.1 Classical 1/8 BPS solutions

As we have described in the introduction, Mikhailov [28] has demonstrated that the intersection of the zero set of the polynomial

$$P(z^i) = \sum_{n_1 n_2 n_3} c_{n_1 n_2 n_3} \mathrm{e}^{-\mathrm{i}(n_1+n_2+n_3)t} (z^1)^{n_1} (z^2)^{n_2} (z^3)^{n_3} \tag{2.1}$$

with the unit 5 sphere $\sum_i |z^i|^2 = 1$ describes the (time dependent) world volume of a $\frac{1}{8}$ BPS giant graviton (see also [34]). The restriction of the full symplectic form to these solutions yields a symplectic form on the manifold of Mikhailov's solutions; the variables $c_{\vec{n}}$ constitute a set of (projective) coordinates on this manifold.

2.2 The symplectic form for spatial motion of D3 branes

The phase space of a classical system may be identified with the space of solutions to the equation of motion of that system. Canonical quantization yields a symplectic form on phase space, and (by restriction) on appropriate sub-classes of solution space [48, 49].

In this section we study the symplectic form on the world volume of $D3$-branes. In particular we study the motion of $D3$-branes in a geometrical space with metric $\widetilde{G}_{\mu\nu}$ and 4 form potential $A_{\mu_1\mu_2\mu_3\mu_4}$, and derive a formal expression for the symplectic form restricted to motions that are purely spatial, i.e. solutions in which the worldvolume field strength $F_{\mu\nu}$ is identically zero.

The action on the world volume of a $D3$-brane is given by

$$\begin{aligned} S &= S_{\mathrm{BI}} + S_{\mathrm{WZ}} \\ &= \frac{1}{(2\pi)^3(\alpha')^2 g_s} \int \mathrm{d}^4\sigma \sqrt{-\tilde{g}} + \int \mathrm{d}^4\sigma \, A_{\mu_0\mu_1\mu_2\mu_3} \frac{\partial x^{\mu_1}}{\partial \sigma^{\alpha_0}} \frac{\partial x^{\mu_1}}{\partial \sigma^{\alpha_1}} \frac{\partial x^{\mu_2}}{\partial \sigma^{\alpha_2}} \frac{\partial x^{\mu_3}}{\partial \sigma^{\alpha_3}} \frac{\epsilon^{\alpha_0\alpha_1\alpha_2\alpha_3}}{4!} \\ &= \frac{1}{(2\pi)^3(\alpha')^2 g_s} \int \mathrm{d}^4\sigma \sqrt{-\tilde{g}} + \int \mathrm{d}t \, \mathrm{d}^3\sigma \, A_{\mu_0\mu_1\mu_2\mu_3} \, \dot{x}^{\mu_0} \frac{\partial x^{\mu_1}}{\partial \sigma^1} \frac{\partial x^{\mu_2}}{\partial \sigma^2} \frac{\partial x^{\mu_3}}{\partial \sigma^3} \, , \end{aligned} \tag{2.2}$$

where

$$\tilde{g}_{\alpha\beta} = \widetilde{G}_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^\alpha}\frac{\partial x^\nu}{\sigma^\beta}\,, \tag{2.3}$$

The symplectic form of interest is given by

$$\begin{aligned}\omega_{\rm full} = \omega_{\rm BI} + \omega_{\rm WZ} &= \int_\Sigma \mathrm{d}^3\sigma\; \delta((p_\mu)_{\rm BI} + (p_\mu)_{\rm WZ}) \wedge \delta x^\mu \qquad (2.4)\\ &= \int_\Sigma \mathrm{d}^3\sigma\; \delta\!\left(\frac{1}{(2\pi)^3(\alpha')^2 g_s}\left(\sqrt{-\tilde{g}}\tilde{g}^{0\alpha}\frac{\partial x^\nu}{\partial\sigma^\alpha}\widetilde{G}_{\mu\nu}\right) + A_{\mu\mu_1\mu_2\mu_3}\frac{\partial x^{\mu_1}}{\partial\sigma^1}\frac{\partial x^{\mu_2}}{\partial\sigma^2}\frac{\partial x^{\mu_3}}{\partial\sigma^3}\right)\wedge\delta x^\mu.\end{aligned}$$

The forms $\omega_{\rm BI}$ and $\omega_{\rm WZ}$ are both manifestly closed. In fact $\omega_{\rm BI}$ is also exact; the same is not true of $\omega_{\rm WZ}$ as the 4 form gauge field $A_{\mu_1\mu_2\mu_3\mu_4}$ is not globally well defined. Equation (2.4) may be massaged into (see appendix C.1)

$$\begin{aligned}\omega_{\rm full} = &\frac{1}{(2\pi)^3(\alpha')^2 g_s}\int_\Sigma \mathrm{d}^3\sigma\; \delta\!\left(\sqrt{-\tilde{g}}\tilde{g}^{0\alpha}\frac{\partial x^\nu}{\partial\sigma^\alpha}\widetilde{G}_{\mu\nu}\right)\wedge\delta x^\mu \\ &+\int_\Sigma \mathrm{d}^3\sigma\;\frac{\delta x^\lambda\wedge\delta x^\mu}{2}\left(\frac{\partial x^\nu}{\partial\sigma^1}\frac{\partial x^\rho}{\partial\sigma^2}\frac{\partial x^\sigma}{\partial\sigma^3}\right)F_{\lambda\mu\nu\rho\sigma}\,,\end{aligned} \tag{2.5}$$

where $F_{\lambda\mu\nu\rho\sigma}$ is the 5 form field strength.

We are interested in the motion of a D3-brane on the S^5 of $AdS_5\times S^5$. In this background the 5-form is given by $F = \frac{2\pi N}{\pi^3}\epsilon$ where ϵ is the volume form on S^5 and π^3 is the total volume of the unit 5-sphere. Moreover $\widetilde{G}_{\mu\nu} = \sqrt{(4\pi(\alpha')^2 g_s N)}G_{\mu\nu}$ where $G_{\mu\nu}$ is the metric on unit radius $AdS_5\times S^5$. Plugging in we find

$$\omega_{\rm full} = \omega_{\rm BI} + \omega_{\rm WZ} = \frac{N}{2\pi^2}\int_\Sigma \mathrm{d}^3\sigma\;\delta\!\left(\sqrt{-g}g^{0\alpha}\frac{\partial x^\nu}{\partial\sigma^\alpha}G_{\mu\nu}\right)\wedge\delta x^\mu \tag{2.6}$$

$$+\frac{2N}{\pi^2}\int_\Sigma \mathrm{d}^3\sigma\;\frac{\delta x^\lambda\wedge\delta x^\mu}{2}\left(\frac{\partial x^\nu}{\partial\sigma^1}\frac{\partial x^\rho}{\partial\sigma^2}\frac{\partial x^\sigma}{\partial\sigma^3}\right)\epsilon_{\lambda\mu\nu\rho\sigma}\,. \tag{2.7}$$

where $g_{\alpha\beta}$ is now defined by

$$g_{\alpha\beta} = G_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^\alpha}\frac{\partial x^\nu}{\partial\sigma^\beta}\,. \tag{2.8}$$

The integrals in (2.4)–(2.6) are all taken over surfaces of constant σ^0; the invariance of the symplectic form under a coordinate redefinition follows from the equations of motion [50].

2.3 Symplectic form on Mikhailov's solutions

In the previous subsection, we have derived a general expression for the symplectic form $\omega_{\rm full} = \omega_{\rm WZ} + \omega_{\rm BI}$ on the space of spatial motions of D3 branes on S^5. While the expression for $\omega_{\rm WZ}$ is simple and geometrical, the expression for $\omega_{\rm BI} = \mathrm{d}\theta_{\rm BI}$ is rather complicated. It turns out to be possible to find a relatively simple geometrical expression for that the restriction of $\theta_{\rm BI}$ to the submanifold of Mikhailov's solutions (see appendix C.2)

$$\begin{aligned}\theta_{\rm BI} &= \frac{N}{2\pi^2}\int_\Sigma \mathrm{d}^3\sigma\;\sqrt{-g}\,g^{0\alpha}\frac{\partial x^\mu}{\partial\sigma^\alpha}\,G_{\mu\nu}\,\delta x^\mu \\ &= \frac{N}{\pi^2}\int_S \mathrm{d}^4\sigma\;\epsilon_{\mu_1\cdots\mu_6}\left[\frac{\partial x^{\mu_1}}{\partial\sigma^1}\cdots\frac{\partial x^{\mu_4}}{\partial\sigma^4}\right]e_\perp^{\mu_5}\,\delta x^{\mu_6}\,\delta\!\left(|z^i|^2-1\right)\,,\end{aligned} \tag{2.9}$$

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where the integral is taken over the four dimensional spatial volume of the holomorphic surface S at constant time and $e_\perp$ is the unit position vector in $\mathbb{C}^3$ (the vector from the origin to the point in question, normalized to have unit norm, so $e_\perp(\underline{w}) = \underline{w}/\|\underline{w}\|$ for all $\underline{w} \in \mathbb{C}^3$).

2.4 Geometrical description of the symplectic form

The contraction of $\omega_{\rm WZ}$ with arbitrary infinitesimal vectors v_1 and v_2 at a threefold Σ in S^5 may be given a simple geometrical interpretation: it is proportional to the 5 volume formed from out of Σ and the two vector fields v_1 and v_2 supported on Σ; the details are given in the next subsection.

Let us now turn to $\theta_{\rm BI}$. The contraction of $\theta_{\rm BI}$ with an infinitesimal vector v_1 is given by the following slightly elaborate geometrical construction. Consider the volume, δV, of formed out of that part of the (real) 4 surface $P(z) = 0$ that intersects a shell of thickness δr around the unit sphere, the unit normal vector field $\mathrm{e}_\perp$, and v_1. The contraction of v_1 with $\theta_{\rm BI}$ is proportional to $\frac{\delta V}{\delta r}$.

We will now reword the constructions above more formally; this will allow us to establish certain smoothness properties of these forms in the next subsection.

We have seen

$$\omega_{\rm full} = \mathrm{d}\theta_{\rm BI} + \omega_{\rm WZ}\,. \tag{2.10}$$

Let $\mathcal{N}$ denote the moduli space of intersections of holomorphic 2 surfaces with S^5 (we will have a lot more to say about $\mathcal{N}$ below). We will now define $\mathcal{M}$, a real codimension 2 real submanifold in $S^5 \times \mathcal{N}$. The submanifold $\mathcal{M}$ consists of all points $(x, z) \in S^5 \times \mathcal{N}$ with the property that the point $x \in S^5$ is in the holomorphic two surface corresponding to z. Therefore, $\mathcal{M}$ is the fibration over $\mathcal{N}$ whose fibre over a point in $\mathcal{N}$ is the intersection of S^5 with the holomorphic 2-surface in question. The dimension of the real manifold $\mathcal{M}$ is $\dim_{\mathbb{R}} \mathcal{N} + 3$.

The volume form ϵ_5 on S^5 has a natural lift to a 5-form on the space $S^5 \times \mathcal{N}$ (i.e. to a 5-form that contracts in the usual way with vectors on S^5, but to zero with vectors on $\mathcal{N}$). Upon restriction this form yields a 5-form on $\mathcal{M}$ (recall that $\mathcal{M}$ is a submanifold of $S^5 \times \mathcal{N}$). Integrating this 5-form over the (generically 3 dimensional) fibres in $\mathcal{M}$ yields a 2-form on $\mathcal{N}$. The two form obtained via this process is proportional to $\omega_{\rm WZ}$.

The form $\theta_{\rm BI}$ has a similar description. We define $\mathcal{M}'$, a real codimension two (complex codimension one) submanifold in $\mathbb{C}^3 \times \mathcal{N}$, as the fibration of those points in $\mathbb{C}^3$ that lie in the holomorphic 2 surface [5] labeled by the base point $\mathcal{N}$.

Consider a (distributional or current) 5 form in $\mathbb{C}^3$ defined by

$$\begin{aligned} \epsilon_5' &= (\iota_{e_\perp}\epsilon_6)\,\psi_\tau(1 - |z|^2)\,, \\ \psi_\tau(x) &= \begin{cases} 1,\ 0 \le x \le \tau, \\ 0,\ \text{otherwise}, \end{cases} \end{aligned} \tag{2.11}$$

[5] More precisely it is the set of points that lie on any holomorphic 2-surface that has the appropriate intersection with S^5; which holomorphic surface is chosen does not matter.

where $e_\perp$ is the unit normal vector in $\mathbb{C}^3$ and the symbol ι denotes contraction of differential forms with a vector field. As above ϵ_5' defines a 5-form on $\mathcal{M}'$ by pulling back. Integrating this 5-form over the 4 real dimensional fibres in $\mathcal{M}'$ produces a one form in $\mathcal{N}$. The form $\theta_{\rm BI}$ is proportional to its derivative with respect to τ at $\tau = 0$.

2.5 Smoothness of $\omega_{\rm WZ}$ and $\theta_{\rm BI}$

We will now use the construction of the previous subsection to demonstrate that the restriction of $\omega_{\rm WZ}$ and $\theta_{\rm BI}$ to any compact, finite (n_C) dimensional subspace $\mathcal{N}_C$ is a 'current'. A current on $\mathcal{N}_C$ is, by definition, a form whose singularities (if any) are mild enough to permit well defined integration against genuine forms of degree $n_C - d$ on $\mathcal{N}_C$. We will show that $\omega_{\rm WZ}$ is a current of degree two and $\theta_{\rm BI}$ is a current of degree one. See chapter 1, section 2 of [47] for a discussion of the properties of currents.

As in the previous subsection, let $\mathcal{M}_C$ denote the (real) codimension 2 real submanifold in $S^5 \times \mathcal{N}_C$ consists of all points $(x, z) \in S^5 \times \mathcal{N}_C$ such that the point $x \in S^5$ is in the holomorphic 2 surface corresponding to z. Consider a smooth $n_C - 2$ form β on $\mathcal{N}_C$. From the definition of $\omega_{\rm WZ}$ we have

$$\int_{\mathcal{N}_C} \omega_{\rm WZ} \wedge \beta \propto \int_{\mathcal{M}_C} f^*(\epsilon_5) \wedge g^*(\beta)\,,$$

where $g : \mathcal{M}_C \to \mathcal{N}_C$ is the projection defined by $(x, z) \mapsto z$, and $f : \mathcal{M}_C \to S^5$ is the projection defined by $(x, z) \mapsto x$. The above identity shows that $\omega_{\rm WZ}$ is a current on $\mathcal{N}_C$ of degree two.

We will now give a similar description of $\theta_{\rm BI}$.

For any $\tau \in (0, 1)$, let

$$S_\tau := \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid 1 - \tau \le |z_1|^2 + |z_2|^2 + |z_3|^2 \le 1\} \subset \mathbb{C}^3$$

be the solid shell in $\mathbb{C}^3$ of width τ. Let $\mathcal{M}_C^\tau$ be the fibration over $\mathcal{N}_C$ of real dimension four defined by all $(x, z) \in S_\tau \times \mathcal{N}$ such that the point x lies in the complex 2 surface (in $\mathbb{C}^3$) corresponding to z.

Take any smooth form β on $\mathcal{N}_C$ of degree $n_C - 1$. It follows from the description of $\theta_{\rm BI}$ in the previous subsection that

$$\int_{\mathcal{N}_C} \theta_{\rm BI} \wedge \beta \propto \frac{\mathrm{d}}{\mathrm{d}\tau} \int_{\mathcal{M}_C^\tau} \phi^*(\iota_{e_\perp}\epsilon_6) \wedge \gamma^*(\beta)\bigg|_{\tau=0}\,, \tag{2.12}$$

where $\phi : \mathcal{M}_C^\tau \to \mathbb{C}^3$ is the projection defined by $(x, z) \mapsto x$ (recall that $\mathcal{M}_C^\tau \subset S_\tau \times \mathcal{N} \subset \mathbb{C}^3 \times \mathcal{N}$), $\gamma : \mathcal{M}_C^\tau \to \mathcal{N}_C$ is the projection defined by $(x, z) \mapsto z$, and $\iota_{e_\perp}\epsilon_6$ is the contraction of the standard volume form ϵ_6 on $\mathbb{C}^3$ by the radial vector field $e_\perp$ on $\mathbb{C}^3 \setminus \{0\}$ that assigns any $x \in \mathbb{C}^3 \setminus \{0\}$ the tangent vector $x/\|x\| \in T_x^{1,0}\mathbb{C}^3 = \mathbb{C}^3$.

From (2.12) it follows immediately that $\theta_{\rm BI}$ is a current on $\mathcal{N}_C$ of degree one.

This conclusion is important for the following reason. As we will see below, the manifold $\mathcal{N}_C$ contains several points of degeneration; for instance points of topology change at which the number of connected components of the holomorphic 2 surface changes. Intuitively one might expect the symplectic form to develop singularities at such points, and in certain coordinate systems this is indeed the case (see appendix D.3). The results of this subsection guarantee that these singularities are tame enough to be dealt with, i.e. to permit geometric quantization, as all of the necessary structures associated with forms can also be defined for currents.

2.6 U(3) charges

Let L^m denote the generators of U(3) on $\mathbb{C}^3$. We choose our basis in the space of U(3) generators such that L^1, L^2, L^3 are the generators corresponding to three $\mathrm{U}(1) \in \mathrm{U}(3)$ charges that generate the rotations $z^i \to \mathrm{e}^{-\mathrm{i}\alpha} z^i$ where $i = 1 \ldots 3$. Let $\xi^{m\alpha}$ denote the infinitesimal variations of the phase space coordinates x^α under the generator L^m. Every symplectic form ω we study in this paper is U(3) invariant; this means that

$$\mathcal{L}_\xi \omega = 0\,, \quad \text{i.e.} \;\; \xi^\alpha \partial_\alpha \omega_{\beta\gamma} + \omega_{\alpha\gamma}\partial_\beta \xi^\alpha + \omega_{\beta\alpha}\partial_\gamma \xi^\alpha = 0\,. \tag{2.13}$$

(Here $\mathcal{L}_\xi$ denotes the Lie derivative with respect to the vector field ξ.) Locally $\omega = \mathrm{d}\theta$. It is always possible to choose θ so that it is also U(3) invariant,[6] and we make this choice in what follows.

The Noether procedure yields the formula

$$L^m = \iota_{\xi^m}\theta = \xi^{m\alpha}\theta_\alpha \tag{2.15}$$

for the conserved charges corresponding to the generators L^m. Notice that[7]

$$\mathrm{d}L^m = -\iota_{\xi^m}\omega\,, \quad \text{i.e.} \;\; \partial_\alpha L^m = \omega_{\alpha\beta}\xi^{m\beta}\,. \tag{2.16}$$

Upon quantization, the functions L^m are promoted to operators. According to the rules of geometric quantization (see appendix A)

$$\hat{L}^m = \omega^{\alpha\beta}\partial_\alpha L^m\,(\mathrm{i}\partial_\beta + \theta_\beta) + L^m \; = -\mathrm{i}\xi^{m\alpha}\partial_\alpha\,.$$

As expected, $\hat{L}^m$ is simply the generator of U(3) acting on functions of the phase space coordinates. Note that the final expression for $\hat{L}^m$ is independent of the symplectic form.

3. The topology and symplectic geometry of phase space

We now turn to a study of the distinct intersections with the unit 5-sphere of the equations $P(z) = 0$ for Polynomials $P(z)$ in P_C labeled by a given set, C, of n_C 3-tuples (n_1, n_2, n_3)

[6] This may be proved as follows. Given any θ such that $\mathrm{d}\theta = \omega$, the quantity θ'

$$\theta' = \int \mathrm{d}U\, \phi_U^*(\theta) \tag{2.14}$$

(where ϕ_U represents the diffeomorphism generated by the U(3) element U, and $\phi_U^*(\eta)$ denotes the pull back of an arbitrary form η under this diffeomorphism) also obeys this equation and moreover is U(3) invariant.

[7] The l.h.s. of (2.16) is a manifestly closed one form; consistency demands the same is true of the r.h.s. . That this is indeed the case follows upon using (2.13).

and defined as

$$P_C = \left\{ P(z^i) = \sum_{\vec{n}\in C} c_{\vec{n}}\,(z^1)^{n_1}(z^2)^{n_2}(z^3)^{n_3} \,\middle|\, c_{\vec{n}} \in \mathbb{C} \right\}. \tag{3.1}$$

Points in the projective space $\mathbb{CP}^{n_C-1}$, whose projective coordinates are the coefficients $c_{\vec{n}}$, label these intersections. This labeling, however, suffers from a flaw; it is many to one. The space of distinct intersections of holomorphic 2 surfaces with S^5 is obtained by performing the appropriate identifications $\mathbb{CP}^{n_C-1}$. In this section we study the phase space obtained from this process.

3.1 A hole in $\mathbb{CP}^{n_C-1}$

Let us first study the subset of $\mathbb{CP}^{n_C-1}$ that labels the empty intersection. On any surface $P(z^i) = \sum_{\vec{n}\in C} c_{\vec{n}}\,(z^1)^{n_1}(z^2)^{n_2}(z^3)^{n_3} = 0$, there will be some points that are nearest to the origin $z^i = 0$. Let us define a distance function $\rho(c,\bar{c})$ to be the distance to these nearest points:

$$\rho(c,\bar{c}) = \min \left\{ \sum_{i=1}^{3} |z^i|^2 \,\middle|\, P(z) = \sum_{\vec{n}\in C} c_{\vec{n}}\,(z^1)^{n_1}(z^2)^{n_2}(z^3)^{n_3} = 0 \right\} \tag{3.2}$$

(below we will sometimes use the alternate symbolic notation $\rho(P(z))$ for (3.2)). The surface will only intersect the sphere if $\rho \leq 1$. Consequently, the set of points $c_{\vec{n}}$ such that $\rho(c,\bar{c}) > 1$ all yield $P(z)$ with the same (namely empty) intersection with the unit 5-sphere. All these points have to be contracted away in our phase space; we will sometimes refer to the set of these points as a hole (in $\mathbb{CP}^{n_C-1}$).

3.2 ω at the boundary of the hole

Consider polynomials $P(z)$ such that $\rho(P(z)) = 1$. Such polynomials lie at the boundary of the hole described above; for these polynomials the surface $P(z) = 0$ skims the unit sphere without cutting. In appendix E, we demonstrate that, when $\rho(P(z)) = 1$, the intersection of any holomorphic surface $P(z) = 0$ with the unit sphere is of real dimension ≤ 2. [8] This fact has immediate implications for the restriction of the symplectic form to the boundary of the hole.

Recall from section 2.4 that the contraction of ω_{WZ} with vectors v_1 and v_2 is proportional to the 5 volume formed from out of the intersection surface, v_1 and v_2. It follows immediately that $\omega_{\mathrm{WZ}} = 0$ when the intersection of $P(z) = 0$ is less than 3 dimensional. In particular, ω_{WZ} vanishes when restricted to the boundary of the hole.

Recall, also from section 2.5, that the current θ_{BI} satisfies the identity in (2.12). From (2.12) it follows that the restriction of θ_{BI} to the boundary vanishes whenever it is defined. We emphasize that θ_{BI} is a current which need not be a smooth differential form, hence it may not be well defined everywhere (see appendix E for more on this).

[8] More generally, for any polynomial P in n variables with $\rho(P(z)) = 1$, the hypersurface $P(z) = 0$ of $\mathbb{C}^n$ intersects the unit sphere on a real surface whose real dimension is at most $n-1$. For example, the surfaces $(z^1)^2 = 1$, $(z^1)^2 + (z^2)^2 = 1$, $\ldots$, $\sum_{i=1}^{n}(z^i)^2 = 1$ touch the unit sphere at a point, a line, $\ldots$, an $n-1$ sphere respectively.

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3.3 Contracting away the hole

We have argued in the previous subsection that $\omega_{\text{full}} = \omega_{\text{WZ}} + d\theta_{\text{BI}}$ vanishes when restricted to the boundary of the hole. This suggests that in the correct coordinates all points within and at the boundary of the hole are identified.[9] Of course this interpretation is consistent if and only if all points in the hole (including the boundary) may continuously be contracted to a single point. We will now show that this is indeed the case.

Given n_C monomials in three variables, there is a hole the corresponding projective space $\mathbb{CP}^{n_C-1}$ formed out of linear combinations of them contains a hole if and only if the constant function 1 is one of the n_C monomials. Assume that 1 is among the n_C monomials.

Notice that the distance function ρ has the following homogeneity property

$$\rho\left(\lambda^{n_1+n_2+n_3} c_{n_1,n_2,n_3}, \lambda^{m_1+m_2+m_3} \bar{c}_{m_1,m_2,m_3}\right) = \lambda^{-1} \rho\left(c_{n_1,n_2,n_3}, \bar{c}_{m_1,m_2,m_3}\right) \tag{3.3}$$

In order to demonstrate that the hole is contractible (in fact it is diffeomorphic to a ball), let $\lambda(t)$ be a decreasing function on $[0,1]$ such that $\lambda(0) = 1$ and $\lambda(1) = 0$. According to (3.3) the map $c_{n_1,n_2,n_3} \to c_{n_1,n_2,n_3} \lambda(t)^{m_1+m_2+m_3}$ continuously maps every point in the hole to its 'center' $c_{\vec{n}} = 0$ (the constant function 1).

It follows immediately that the space obtained by contracting away the hole is $\mathbb{CP}^{n_C-1}$ itself. We will now demonstrate this (intuitively obvious) fact. In order to identify the space obtained after this contraction, let $h(\rho)$ be any nondecreasing function defined on $[0, \infty)$ satisfying $h(0) = 1$ and $h(\rho) = 0$ for $\rho \geq 1$. The coordinate change

$$w_{\vec{n}} = [h(\rho)]^{n_1+n_2+n_3} c_{\vec{n}} \tag{3.4}$$

yields a continuous map from $\mathbb{CP}^{n_C-1}$ minus the hole to $\mathbb{CP}^{n_C-1}$, the last space being parameterized by the w coordinates. Notice that this map takes all points in the hole, including the boundary $\rho = 1$ to a single point, the 'origin' in the new space. It follows that our original space, with the hole shrunk to a point is topologically $\mathbb{CP}^{n_C-1}$.

The distinguished new point — the 'origin' in w coordinates represents all configurations that graze the unit ball together with those configurations that do not intersect the unit ball at all. All of these a priori distinct physical configurations map to the same point in physical phase space.

3.4 Characterization of distinct intersections

Now let us turn to a consideration of manifolds $P(z) = 0$ that cut (and don't just graze) the unit S^5, i.e. polynomials for which $\rho(P(z)) < 1$.

Suppose $S = \{z \in \mathbb{C}^3 | f(z) = 0\}$ and $T = \{z \in \mathbb{C}^3 | g(z) = 0\}$ are two irreducible hypersurfaces (this means that neither of f and g factorize) such that both S and T cut the unit sphere S^5 (in $\mathbb{C}^3$), and also, $S \bigcap S^5 = T \bigcap S^5$. Note that $S \cap T$ is a complex variety. Since the real dimension of $S \bigcap S^5$ is 3 (recall that S cuts S^5 and not just grazes it), any

[9] As an analogy, flat d dimensional space, when written in polar coordinates, appears to have a boundary S^{d-1} at $r = 0$. However the vanishing of the metric restricted to the S^{d-1} is a clue that all of the 'boundary' is in fact a single bulk point. We will see a better example of this phenomenon – involving symplectic forms rather than metrics — in section 5.

complex variety containing $S \bigcap S^5$ must be of complex dimension at least two. Since the complex dimensions of S and T are two, these imply that S and T share a common open subset. Given that S and T are both irreducible, from this it follows that $S = T$. Hence $f = c.g$, where c is some nonzero complex number.

Therefore, two polynomials $P_1(z) = 0$ and $P_2(z) = 0$ have identical, three dimensional, intersections with the unit 5-sphere if and only if $P_1(z) = q(z)r_1(z)$ and $P_2(z) = q(z)r_2(z)$ where $r_1(z)$ and $r_2(z)$ are polynomials with distance functions ≥ 1.

Thus physical phase space — the phase space of distinct intersections of $P(z) = 0$ with the unit 5-sphere — is the $\mathbb{CP}^{n_C - 1}$ space parameterized by polynomial coefficients, subject to the following identification: polynomials of the form $f(z)g(z)$ are to be identified with $f(z)$ when $\rho(g(z)) \geq 1$.

3.5 The topology of distinct intersections

We will now argue that the space described in the previous subsection is, in fact, diffeomorphic to $\mathbb{CP}^{n_C - 1}$. This result follows almost immediately from general considerations that we now briefly review.

Consider a smooth real manifold M of dimension ℓ. Let S be a closed submanifold of M, and let $f : S \to Q$ be a smooth projection, such that each fibre of the projection f is diffeomorphic to the unit ball in $\mathbb{R}^\ell$ (we note that S is allowed to have dimensions smaller than that of M). Then it is always possible to find a map D from $M \times [0, 1]$ to M such that $D(-, t)$ is a diffeomorphism on M for t in $(0, 1)$ with $D(-, 0)$ being the identity map of M, and furthermore, $D(-, 1)$ reduces to a contraction when restricted to any of the fibres of of the projection f; see [51, theorem 5.8]. This result may be worded more pithily; a manifold M retains its diffeomorphism type upon contracting away any embedded family of balls of arbitrary dimension; the contraction is done along the direct of the balls, or in other words, each individual ball in the family is contracted to a single point, but distinct balls are contracted to distinct points. If there are finitely many disjoint embedded fibre bundles of the above type, then we contract one by one (of this finite collection). After each step of the contraction, the resulting manifold remains diffeomorphic to the one in previous step. Therefore, the final manifold remains diffeomorphic to the original one.

We will see how these results apply to our situation in an example. Let P_C denote the set of polynomials of degree at most two; this space is some contraction of $\mathbb{CP}^9$. The only new identification (apart from the hole in this space) is $p(z)q(z) \sim p(z)$ where p and q are each degree one polynomials, and $\rho(q(z)) \geq 1$. Contracting away the hole in $\mathbb{CP}^9$ already identifies all polynomials $p(z)q(z)$ where $\rho(p) \geq 1$ and $\rho(q) \geq 1$ with a single point (the 'origin' in $\mathbb{CP}^9$ which corresponds to the constant function 1). The arguments of subsection 3.3 establish that the set of points in $\mathbb{CP}^9$ corresponding to polynomials $p(z)q(z)$ with a fixed $p(z)$ but varying over all $q(z)$ such that $\rho(q(z)) \geq 1$, is contractible, in fact it is diffeomorphic to a ball in a Euclidean space. Associate this set, which is diffeomorphic to a ball, to $p(z)$. Now varying over all $p(z)$ with $\rho(p(z)) < 1$ gives a family of disjoint balls. According to the theorem quoted at the beginning of this subsection, the space obtained after contracting away these sets continues to have the topology of $\mathbb{CP}^9$.

In the general case one may proceed similarly, first shrinking away the hole, then dealing with polynomials with 2, 3, ... factors. At every stage in this process we always contract away disjoint sets of balls (of arbitrary dimensions), and so the theorem quoted above guarantees that the space we obtain at the end of this whole process is $\mathbb{CP}^{n_C-1}$. This $\mathbb{CP}^{n_C-1}$ space may parameterized by a set of projective coordinates $w_{\vec{n}}$ where $\vec{n}$ belongs to the set C. The coordinates $w_{\vec{n}}$ may be thought of as functions of the polynomial coefficients $c_{\vec{n}}$. Since each component that is contracted to a point is left invariant by the standard action of $\mathrm{U}(3)$ on $\mathbb{CP}^{n_C-1}$, the quotient space (the space obtained after all the contractions are done) is equipped with an action of $\mathrm{U}(3)$ which is induced by the action of $\mathrm{U}(3)$ on $\mathbb{CP}^{n_C-1}$.

As the distance function ρ is $\mathrm{U}(3)$ invariant, it is possible to perform all contractions in a $\mathrm{U}(3)$ invariant manner (this can be checked for the variable change (3.4) that contracts away the hole) [10] so that

$$w_{\vec{n}} = c_{\vec{n}} f_{|n|}(c, \bar{c}) \tag{3.5}$$

where $f_{|n|}$ are $\mathrm{U}(3)$ invariant functions and $|n| \equiv \sum_i n_i$.

3.6 Cohomology class of the symplectic form

Recall from section 2.4 that ω_{WZ} is the fibre integral of a closed 5 form, and so is closed. The form ω_{BI} is also closed (it is exact). It is known that $H^2(\mathbb{CP}^m, \mathbb{Z}) = \mathbb{Z}$ for all $m > 0$ (the generator 1 in $H^2(\mathbb{CP}^m, \mathbb{Z})$ is given by the Poincaré dual of a linear hyperplane of (complex) codimension one in $\mathbb{CP}^m$). As before, let ω_{FS} be the usual Fubini-Study form on $\mathbb{CP}^{n_C-1}$. The cohomology class $[\omega_{\mathrm{FS}}] \in H^2(\mathbb{CP}^m, \mathbb{C})$ of the closed form ω_{FS} coincides with $1 \in H^2(\mathbb{CP}^m, \mathbb{Z}) \subset H^2(\mathbb{CP}^m, \mathbb{C})$. Therefore, $[\omega] = M_C[\omega_{\mathrm{FS}}]$ for some complex number M_C, and the equality holds within the second cohomology.

We will now demonstrate that the number M_C is independent of the set C. Consider any two sets of 3-tuples, C_1 and C_2, such that $C_1 \subset C_2$. In the auxiliary $\mathbb{CP}^{n_C-1}$ space (from which physical phase space may be obtained by performing appropriate contractions) the restriction from C_2 to C_1 is very simple; it is achieved by setting to zero the projective coordinates $c_{\vec{n}}$ for those $\vec{n}$ that belong to C_2 but not to C_1. However, under this restriction, the Fubini-Study form in $\mathbb{CP}^{n_{C_2}-1}$ gives the Fubini-Study form in $\mathbb{CP}^{n_{C_1}-1}$. The same is true of cohomology classes. In other words, the homomorphism

$$\mathbb{Z} = H^2(\mathbb{CP}^{n_{C_2}-1}, \mathbb{Z}) \longrightarrow H^2(\mathbb{CP}^{n_{C_1}-1}, \mathbb{Z}) = \mathbb{Z}\,,$$

induced by the inclusion map $\mathbb{CP}^{n_{C_1}-1} \hookrightarrow \mathbb{CP}^{n_{C_2}-1}$, is the identity map of $\mathbb{Z}$. Therefore, the constant M_C may be determined from the study of a single convenient example for C. In section 5 we will study the example $C_0 = (0,0,0), (1,0,0), (0,1,0), (0,0,1)$ in extensive detail, and will find $M_{C_0} = 2\pi N$ (where N is the rank of the gauge group of the dual gauge theory, or the number of units of flux through the S^5). Now, given any C, we define $C' = C \cup C_0$. The restrictions of C' to C and C_0 respectively yield the equations $M_{C'} = M_C$ and $M_{C'} = 2\pi N$; it follows that $M_C = M_{C_0} = 2\pi N$ for every C. This result was anticipated in [29] using physical reasoning.

[10] In this paragraph we assume that the set C consists of full $\mathrm{U}(3)$ multiplets. If this is not the case, the statements in this paragraph continue to hold with $\mathrm{U}(3)$ replaced by $\mathrm{U}(1)^3$

As the map from the auxiliary phase space $\mathbb{CP}^{n_C-1}$ to physical phase space is continuous, it follows that the symplectic form is in the cohomology class of $(2\pi N)\omega_{\rm FS}$ on physical phase space as well.

4. Quantization of phase space

4.1 Quantization

In the previous section we have argued that the phase space $\mathcal{N}_C$ is $\mathbb{CP}^{n_C-1}$ equipped with a symplectic form in the cohomology class of $(2\pi N)\omega_{\rm FS}$. The projective coordinates of this space, $w_{\vec{n}}$, transform under the $\mathrm{U}(1)^3$ transformation $z^i \to \mathrm{e}^{-\mathrm{i}\alpha_i} z^i$ as $w_{\vec{n}} \to \mathrm{e}^{\mathrm{i}n_i\alpha_i} w_{\vec{n}}$.

The Hilbert Space that follows from geometric quantization is the space of polarized sections of the symplectic line bundle (the line bundle whose curvature is the symplectic form). To get our discussion started, let us first suppose that the symplectic form $\omega_{\rm full}$ is $(1,1)$ with respect to the $w_{\vec{n}}, \overline{w}_{\vec{n}}$ complex structure. We have derived the resultant Hilbert Space in appendix A.4; we briefly review the logic here. When $\omega_{\rm full}$ is $(1,1)$ it is possible to choose complex polarization $D_{\overline{w}_{\vec{n}}}\phi = 0$ and to choose the symplectic potential such that $\theta_{\overline{w}_{\vec{n}}} = 0$ (such a potential is said to be adapted to our polarization); the polarization condition is solved when ϕ is an analytic function of $w_{\vec{n}}$; further ϕ is forced to be a function of degree N in the coordinates $w_{\vec{n}}$ in order that it be a globally well defined section. In summary the Hilbert Space $\mathcal{H}_C$ may be identified with the space of degree N Holomorphic Polynomials of $w_{\vec{n}}$. [11] The 3 $\mathrm{U}(1)$ charges L^m are implemented by the operator $L^m = n^m w_{\vec{n}} \partial_{w_{\vec{n}}}$.

The partition function is

$$\mathrm{Tr}_{\mathcal{H}_C} \mathrm{e}^{-\beta_m L^m} = Z_N^C(\beta_m) \tag{4.1}$$

where

$$\sum_{r=0}^{\infty} Z_r^C(\beta_m) p^r = \prod_{\vec{n}\in C} \frac{1}{1 - p\,\mathrm{e}^{-n_m\beta_m}} \tag{4.2}$$

In particular the total number of states in this Hilbert Space is $\binom{n_C-1}{N}$.

The Hilbert space $\mathcal{H}_C$ is isomorphic to another Hilbert space that is more familiar to most physicists. Consider a collection of N identical, noninteracting bosons whose single particle Hilbert Space consists of the vectors $|\vec{n}\rangle$ whose $\mathrm{U}(1)^3$ charges are $L^m|\vec{n}\rangle = n^m|\vec{n}\rangle$. [12] The Hilbert Space of this system is isomorphic to $\mathcal{H}_C$; the function $\prod_{\vec{n}\in C}(w_{\vec{n}})^{m_{\vec{n}}}$ maps to the state in which the occupation number of the single-particle state $|\vec{n}\rangle$ is $m_{\vec{n}}$. Equation (4.1) applies equally well to the partition function over the Hilbert space of these noninteracting bosons.

Equation (4.1) was derived under the assumption that the $\omega_{\rm full}$ is $(1,1)$. The prequantum line bundle is not changed by the addition of an exact form $\mathrm{d}\psi$ to $\omega_{\rm full}$. However

[11] This result is very close to Beasley's conjecture for a Hilbert Space consisting of degree N homogeneous polynomials of the variables $c_{\vec{n}}$.

[12] Note in particular that when C is chosen to be maximal, this single particle Hilbert space is that of the 3d harmonic oscillator, with L^m equal to the number operator of the $m^{\rm th}$ oscillator.

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ψ does constrain the choice of polarization. In particular, if the full symplectic form were not (1,1) we would not be allowed to choose a holomorphic polarization. As a consequence the derivation of (4.1) outlined above does not go through unchanged for arbitrary $\omega_{\rm full}$; nonetheless the final result (4.1) continues to apply as we now demonstrate.

Recall that the infinitesimal change of a closed 2 form ω under an infinitesimal coordinate transformation parameterized by the vector field ζ is equal to the Lie derivative

$$\mathcal{L}_\zeta \omega = \mathrm{d}(\iota_\zeta \omega) + \iota_\zeta \mathrm{d}(\omega) = \mathrm{d}(\iota_\zeta \omega) \ .$$

As ω is invertible, it follows that any infinitesimal deformation, $\mathrm{d}\psi$, of a closed two form that is also exact may be undone by a coordinate change. Denoting the coordinates by x^α the coordinate change in question is given in components by

$$\delta x^\alpha = \omega^{\alpha\beta}\psi_\beta \, .$$

Now suppose ω is $(1,1)$ and ψ is infinitesimal but otherwise arbitrary. It follows from the paragraph above that the Hilbert Space obtained by quantizing $\omega + \mathrm{d}\psi$ with holomorphic polarization in the new coordinates is identical to the Hilbert Space obtained from quantizing ω with holomorphic polarization in the old coordinates. This argument may be iterated step by step by integration and so it holds also for finite ψ. As $\omega_{\rm full}$ and the one form ψ in our case are each $\mathrm{U}(3)$ invariant, the new coordinates have the same $\mathrm{U}(3)$ transformation properties as the old coordinates, and (4.1) applies to all $\omega_{\rm full}$ in the cohomology class of $(2\pi N)\omega_{\rm FS}$.

Notice that we have derived (4.1) knowing only the cohomology class and $\mathrm{U}(3)$ invariance of the symplectic form; in particular, our result was insensitive to the detailed form of $\omega_{\rm BI}$ (which is an exact form, see [29] for related remarks). In that sense, our quantization is analogous to the quantization of the lowest Landau level, as in [52 – 54].

4.2 Semiclassical quantization

Consider any set of polynomials, P_C, as described in (3.1): those built from monomials $(z^1)^{n_1}(z^2)^{n_2}(z^3)^{n_3}$ with (n_1, n_2, n_3) in a given set, C, of 3-tuples. As a check of the slightly formal arguments that have led to (4.1), in this section we directly quantize the intersections of $P = 0$ (for $P \in P_C$) with the unit 5-sphere, in the semiclassical approximation. We demonstrate that the density of states graded by the three $\mathrm{U}(3)$ Cartan charges in this system, to leading order in the effective Plank constant $\frac{1}{N}$, is identical to the density of states obtained, in the same approximation, from the quantization of $\mathbb{CP}^{n_C-1}$ with symplectic form $(2\pi N)\omega_{\rm FS}$ and with $\mathrm{U}(1)^3$ charge operators as described in the previous subsection.

In section 3.6 we saw that, in the auxiliary $\mathbb{CP}^{n_C-1}$ with the $c_{\vec{n}}$s as projective coordinates, the symplectic form $\omega_{\rm full}$ is in the same cohomology class as $(2\pi N)\omega_{\rm FS}$, i.e. we can write

$$(2\pi N)\omega_{\rm FS} - \omega_{\rm full} = \mathrm{d}\psi \, . \tag{4.3}$$

where ψ is a well defined 1-form current on $\mathbb{CP}^{n_C-1}$. Using the transformation in (2.14), we can choose to have a invariant under $\mathrm{U}(1)^3$ (i.e. have vanishing Lie derivative with respect to ξ^m, the vector fields that generate $\mathrm{U}(1)^3$)

Now consider the one parameter set of closed, $\mathrm{U}(1)$ invariant differential forms

$$\omega(x) = \omega_{\text{full}} + \mathrm{d}(x\,\psi)\,. \tag{4.4}$$

Following (2.15), we should also define

$$L^m(x) = L_0^m + x\,\iota_{\xi^m}\psi\,, \tag{4.5}$$

with L_0^m the original Noether charges. With this definition, (2.16) holds for all x. Note that $\omega(0) = \omega_{\text{full}}$ and that $\omega(1) = (2\pi N)\omega_{\text{FS}}$; similar relations hold for $L^m(x)$.

We will now demonstrate that the classical partition function,

$$Z(\beta_i) = \int \frac{\omega(x)^{n_C-1}}{(n_C-1)!(2\pi)^{n_C-1}}\,\mathrm{e}^{-\beta_m L^m(x)}\,, \tag{4.6}$$

is independent of x. Differentiating with respect to x, we find:

$$\begin{aligned}\frac{\mathrm{d}Z}{\mathrm{d}x} &\propto \int \big((n_C-1)\omega(x)^{n_C-2}\wedge \mathrm{d}\psi - \beta_m\omega(x)^{n_C-1}\iota_{\xi^m}\psi\big)\,\mathrm{e}^{-\beta_m L^m(x)} \\ &= \int \Big\{\mathrm{d}\Big[(n_C-1)\omega(x)^{n_C-2}\wedge\psi\,\mathrm{e}^{-\beta_m L^m(x)}\Big] - \beta_m\iota_{\xi^m}\big[\omega(x)^{n_C-1}\wedge\psi\big]\,\mathrm{e}^{-\beta_m L^m(x)}\Big\}\,.\end{aligned} \tag{4.7}$$

The first term vanishes because we are in a compact space with no boundary. The second term vanishes because ω^{n_C-1} is a top form.

We have thus demonstrated that the two symplectic forms ω_{full} and $(2\pi N)\omega_{\text{FS}}$ generate identical densities of states, graded with respect to the $\mathrm{U}(3)$ Cartan charges, in the semiclassical approximation. In particular, Taking the $\beta_i \to 0$ limit, we find that the total number of states, Ω, in Bohr-Sommerfeld quantization is equal to

$$\Omega = \frac{N^{n_C-1}}{(n_C-1)!}\,. \tag{4.8}$$

Note that

$$\lim_{N\to\infty} \frac{\Omega}{\frac{(n_C-1+N)!}{(n_C-1)!N!}} = 1\,. \tag{4.9}$$

Consequently, in the large N (semiclassical) limit Ω reproduces the exact number of states, $\frac{(n_C-1+N)!}{(n_C-1)!N!}$, that results from the Kähler quantization of $\mathbb{CP}^{n_C-1}$ with symplectic form $(2\pi N)\omega_{\text{FS}}$.

5. Linear polynomials

5.1 Preview of the rest of the paper

The partition function (4.1), the main result of this paper, applies to the quantization of the intersections of $P(z) = 0$ with the unit sphere, for the set of polynomials $P(z) \in P_C$, where C is an arbitrary collection of 3 tuples, and P_C denotes the corresponding linear set of polynomials. As very general arguments that led to (4.1) have been slightly formal in character, we devote the rest of the paper to a more explicit and detailed study of the quantization of special linear sets of polynomials P_C.

In this section we work through the quantization, in gory and explicit detail, for C chosen as the set $(0,0,0),(1,0,0),(0,1,0),(0,0,1)$; i.e. for P_C chosen as the set of polynomials of degree not larger than one. Our results, are all fully consistent with results of the previous sections, and illuminate aspects of the arguments presented above. Our explicit results also allow us determine the coefficient of the symplectic form (this was a loose end in section 3.6).

In section 6 we choose C to be the collection $(m,0,0)$ for $m = 0,1,2\ldots k$. Physically, the polynomials generated by this set of C describe the motion of up to k identical half BPS giant gravitons. Using this interpretation we are able to independently compute the partition function over $\mathcal{H}_C$ in this case. We find agreement with (4.1); we view this as a nontrivial consistency check of (4.1).

In section 7 we study the quantization of homogeneous polynomials of degree k; (i.e. for C chosen as the set (n_1,n_2,n_3) such that $\sum_i n_i = k$). The analysis of 7 uses the formal arguments of the same flavour as those used in sections 3 and 4.1; however the relative simplicity of the space of homogeneous polynomials permits us to be more explicit, and to verify the result (4.1) with a greater degree of mathematical rigour.

5.2 Linear polynomials: setting up the problem

In this section we study the quantization of the linear polynomials

$$c_i z^i - 1 = 0\,, \tag{5.1}$$

where $i = 1\ldots 3$. In order to perform this quantization we need the symplectic (2.6) restricted to Polynomials of the form (5.1). This form must respect U(3) invariance and must also be closed; these conditions constrain the symplectic form to be of the form

$$\omega = f\left(|c|^2\right)\frac{\mathrm{d}\bar{c}^i\wedge\mathrm{d}c_i}{2\mathrm{i}} + f'\left(|c|^2\right)\bar{c}^i c_j\frac{\mathrm{d}\bar{c}^j\wedge\mathrm{d}c_i}{2\mathrm{i}}\,, \tag{5.2}$$

for some function $f(|c|^2)$. In appendix F we have explicitly evaluated contribution of the Wess-Zumino and the Born-Infeld terms to the symplectic form (2.6) (restricted to linear polynomials). Our results are of the form (5.2) with

$$\begin{aligned}
f_{\mathrm{BI}}(|c|^2) &= 2N\left(\frac{1}{|c|^4}-\frac{1}{|c|^6}\right),\\
f_{\mathrm{WZ}}(|c|^2) &= 2N\left(\frac{1}{|c|^2}-\frac{2}{|c|^4}+\frac{1}{|c|^6}\right),\\
f_{\mathrm{full}}(|c|^2) &= 2N\left(\frac{1}{|c|^2}-\frac{1}{|c|^4}\right).
\end{aligned} \tag{5.3}$$

Further, (with no summation over the index m)

$$\omega_{\bar{\imath}j}(\mathrm{i}c^m\delta^j_m) = \frac{1}{2}fc^m\delta^i_m + \frac{1}{2}f'c^i|c^m|^2 = \frac{1}{2}\partial_{\bar{c}_i}\left(|c^m|^2 f(|c|^2)\right)\,. \tag{5.4}$$

Comparing with (2.16) we identify the Noether charges:

$$L^m = \frac{1}{2}|c^m|^2 f\,. \tag{5.5}$$

Notice that $L^1 + L^2 + L^3$ evaluated using $f_{\rm full}$ and (5.5) yields (F.2) as expected on general grounds.

5.3 Semiclassical quantization

We will now discuss the semiclassical quantization of the space of linear polynomials, with respect to the three symplectic form $\omega_{\rm BI}$, $\omega_{\rm WZ}$ and $\omega_{\rm full}$ and associated charges. The Bohr-Sommerfeld density of states $\frac{\omega\wedge\omega\wedge\omega}{3!(2\pi)^3}$ is easily evaluated; we find [13]

$$\int \frac{\omega\wedge\omega\wedge\omega}{3!(2\pi)^3} = \frac{N^3}{2}\int E(x)^2 E'(x)\mathrm{d}x = \frac{N^3}{2}\int E^2\mathrm{d}E = N^3\left(\frac{E^3(\infty)}{6} - \frac{E^3(1)}{6}\right), \tag{5.6}$$

where

$$E(x) = \frac{1}{2N}xf(x)\,, \qquad x = |c|^2\,, \tag{5.7}$$

is the Noether charge $L^1 + L^2 + L^3$. Substituting in formulae for $f(x)$ we conclude that semiclassical quantization with $\omega_{\rm WZ}$ and $\omega_{\rm full}$ each yield a Hilbert space with $N^3/6$ expectations from and density of states (graded with respect to E) proportional to E^2; in perfect agreement with the results of 4.2. Note that the semiclassical quantization using $\omega_{\rm BI}$ yields no states; this is a direct consequence of the fact that $\omega_{\rm BI}$ is exact.

5.4 Exact quantization

In this subsection we will proceed to perform an exact quantization of Mikhailov's linear polynomials. Notice that polynomials with coefficients s.t. $|c|^2 < 1$ fail to intersect the unit sphere; this is the hole of section 3.1. Notice also that $\omega_{\rm full}$, $\omega_{\rm BI}$ and $\omega_{\rm WZ}$ all vanish on the boundary of the hole (see (5.3)), as expected from section 3.2. According to section 3.3 we should deal with the hole by finding a U(3) invariant coordinate change that maps $|c^2| = 1$ to the origin of the new coordinate system. Under a U(3) invariant variable change (i.e. a variable change of the form $c^i = w^i g(|w|^2)$ for any function g), (5.2) retains its form, with

[13] The algebra leading to this result may be processed as follows. Define

$$\alpha = \frac{\mathrm{d}\bar{c}^i\wedge\mathrm{d}c_i}{2\mathrm{i}}\,, \qquad \beta = \frac{\bar{c}^i c_j}{|c|^2}\frac{\mathrm{d}\bar{c}^j\wedge\mathrm{d}c_i}{2\mathrm{i}}\,,$$

and use the identities:

$$\beta\wedge\beta = 0\,, \quad 3\alpha\wedge\alpha\wedge\beta = \alpha\wedge\alpha\wedge\alpha = 6\epsilon_6\,, \quad (a\alpha + b\beta)^3 = 6a^2(a+b)\epsilon_6\,,$$

(where ϵ_6 is the usual volume form on R^6). Substitute $a = f, \quad b = |c|^2 f'$, and $\epsilon_6 = \frac{\Omega_5}{2}|c|^4\mathrm{d}(|c|^2)$ and use the variable x for $|c|^2$.

the function $f(|c|^2)$ being replaced by a new function $\tilde{f}(|w|^2)$ defined by the equation[14]

$$|w|^2 \tilde{f}(|w|^2) = |c|^2 f(|c|^2)\,. \tag{5.9}$$

We will choose our coordinate w to satisfy the equation

$$2N\frac{|w|^2}{1+|w|^2} = |c|^2 f(|c|^2)\,. \tag{5.10}$$

We choose to work in the variable w defined by (5.10) so that the in the new w variables the symplectic form is simply $2\pi N$ times the Fubini-Study form on $\mathbb{CP}^3$ (in the gauge in which one of the coordinates is set to unity)! [15]

Provided that $|c|^2 f(|c|^2)$ increases monotonically from zero at $|c| = 1$ to $2N$ at $|c| = \infty$, the coordinate change defined by (5.10) is legal; i.e. it is one one and maps the region $|c| > 1$ to $\mathbb{C}^3$; the 5 sphere $|c| = 1$ to the origin. In other words, provided $|c|^2 f(|c|^2)$ decreases monotonically between the two limits of interest, (this is true of $\omega_{\rm full}$ as well as $\omega_{\rm WZ}$ but not of $\omega_{\rm BI}$) [16], the space (5.2) is simply $\mathbb{CP}^3$ with symplectic form equal to $2\pi N$ times the Fubini-Study form, written in perverse coordinates, exactly as anticipated in section 3.3, section 3.5 and section 3.6.

The quantization of the space of intersection of linear polynomials with the S^5, with symplectic form given either by $\omega_{\rm WZ}$ or $\omega_{\rm full}$ is now straightforward (see section 4.1 and appendix A). The Hilbert space is given holomorphic polynomials, of degree N, of the four variables $1, w^1, w^2, w^3$. On this Hilbert space the charge operators are simply $L^1 = w^1\partial_{w^1}$, $L^2 = w^2\partial_{w^2}$, $L^3 = w^3\partial_{w^3}$. As explained in the introduction, this is precisely the Hilbert space of a system of N identical noninteracting bosons, whose single particle Hilbert space consists of four states with charges (L^1, L^2, L^3) equal to $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

The partition function over this Hilbert Space is, of course, given by (4.1) with the choice of C listed above. In particular upon setting $\beta_1 = \beta_2 = \beta_3 = \beta$ we find

$$Z_{\rm Lin} = \sum_{k=0}^{N} \binom{k+2}{2} e^{-\beta k} \tag{5.12}$$

[14] More explicitly

$$\tilde{f}(|w|^2) = \left|\frac{c(w)}{w}\right|^2 f(|c(w)|^2) = \left|g(|w|^2)\right|^2 f\Big(|w|^2\left|g(|w|^2)\right|^2\Big)\,. \tag{5.8}$$

[15] Note that (5.10) ensures that $\tilde{f}(w^2) = 2N\frac{1}{1+w^2}$; plugging into (5.2) ω turns into $2\pi N$ times the Fubini-Study form.

[16] Explicitly, the variable change for $\omega_{\rm full}$ is

$$c_i = \sqrt{\frac{|w|^2+1}{|w|^2}}\, w_i\,, \tag{5.11}$$

i.e. choosing $h(\rho) = \sqrt{1-\rho^2}$ in (3.4).

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In the large N limit the summation over k may be replaced by an integral over a continuous variable E and we find

$$Z_{\rm Lin} \approx \int_0^N \mathrm{d}E \, \frac{E^2}{2} \, \mathrm{e}^{-\beta E} \tag{5.13}$$

in perfect agreement with the semiclassical results of section 5.3.

6. Polynomials of a single variable

6.1 Expectations from section 4.1

In this subsection we study the quantization of degree k polynomials of a single variable z. According to the general arguments of section 4.1, this submanifold of solutions should be topologically $\mathbb{CP}^k$; the Hilbert space obtained by quantizing these solutions should be given by degree N polynomials of the variables v_0, $v_1 \ldots v_k$ respectively where the subscript denotes the L^3 charges of these variables. The partition function $\mathrm{Tr}\ \mathrm{e}^{-\beta L^3}$ over this Hilbert space is predicted to be (4.1) $\widetilde{Z}_N$ where

$$\sum_{m=1}^{\infty} q^m \widetilde{Z}_m = \prod_{r=0}^{k} \frac{1}{1 - q\mathrm{e}^{-\beta r}} \, . \tag{6.1}$$

6.2 Direct evaluation

We will now test all these predictions against an independent direct study of this space and its quantization. Any degree k polynomial of a single variable is proportional to

$$P(z) = \prod_{a=1}^{k} \left(\frac{\sqrt{1+|w^a|^2}}{|w^a|} \, w^a z - 1 \right) . \tag{6.2}$$

for some choice of w^a. The intersection of $P(z) = 0$ with the unit sphere describes a gas of noninteracting half BPS giant gravitons; i.e. the corresponding D3-brane surfaces consist of a set of disconnected, parallel S^3s of squared radius $\frac{|w^a|^2}{1+|w^a|^2}$. [17]

The $\mathrm{U}(1)$ charges of these branes are additive. Geometrically the space parameterized by the coordinates w^a is $(\mathbb{CP}^1)^k/S_k$ (where S_k is the permutation group) and the symplectic form on this space is

$$\omega = \sum_a \omega_a \, , \tag{6.3}$$

where

$$\omega_a = \frac{2N}{1+|w^a|^2} \left(\frac{\mathrm{d}\overline{w}^a \wedge \mathrm{d}w^a}{2\mathrm{i}} - \frac{w^a \overline{w}^a}{1+|w^a|^2} \frac{\mathrm{d}\overline{w}^a \wedge \mathrm{d}w^a}{2\mathrm{i}} \right) \tag{6.4}$$

is $2\pi N$ times the Fubini-Study form on $\mathbb{CP}^1$.

The Hilbert space obtained from the quantization of (6.4) is the set of symmetric holomorphic polynomials of degree N or less in each variable w^a (the symmetry of polynomials

[17] Half BPS giant gravitons have been studied extensively, see for instance [55 – 63, 30, 64, 31, 32, 65, 66]. The discussion below overlaps with these studies.

is forced from the fact that the polynomials $P(z)$ in (6.2) are invariant under permutations of w^a). The partition function weighted by $e^{-\beta H}$ (where H is the U(1) rotation of z) over this Hilbert space is given by Z_k where

$$\sum_k p^k Z_k = \prod_{m=0}^{N} \frac{1}{1 - p\mathrm{e}^{-\beta m}} \,. \tag{6.5}$$

6.3 Comparison

We will now demonstrate that the geometrical spaces and Hilbert Spaces described in subsections 6.1 and 6.2 are the same.

The quotient space $(\mathbb{CP}^1)^k/S_k$ may be holomorphically identified with $\mathbb{CP}^k$ in the following way: Let (x^a, y^a) be projective coordinates for the a^{th} $\mathbb{CP}^1$. Let z^j $(j = 0, \dots, k)$ be projective coordinates for $\mathbb{CP}^k$. Let S_j be the set of subsets of $\{1, \dots, k\}$ of cardinality j. The map

$$z^j = \sum_{\tau \in S_j} \left[\prod_{a \in \tau} x^a \prod_{b \in \tau^c} y^b \right]$$

provides the identification between $(\mathbb{CP}^1)^k/S_k$ and $\mathbb{CP}^k$. Furthermore, if the x^a's have charge 1 and the y^a's have charge 0, then z^j will have charge j.

The cohomology class of $(\mathbb{CP}^1)^k/S_k$ defined by ω coincides with the cohomology class of $\mathbb{CP}^k$ given by the Fubini-Study form on $\mathbb{CP}^k$ (after identifying $(\mathbb{CP}^1)^k/S_k$ with $\mathbb{CP}^k$). Indeed, this follows from the facts that the degree of the projection $(\mathbb{CP}^1)^k \to (\mathbb{CP}^1)^k/S_k$ is $k!$, and

$$\int_{(\mathbb{CP}^1)^k} \omega^k = k! \,.$$

All of this implies that the Hilbert Space obtained from the quantization of $\frac{(\mathbb{CP}^1)^k}{S_k}$ is isomorphic to that obtained from the holomorphic quantization of $\mathbb{CP}^k$ whose projective coordinates have charges $(0, 1, \dots, k)$; i.e. implies that

$$\sum_{N=0}^{\infty} q^N \prod_{m=0}^{N} \frac{1}{1 - p\mathrm{e}^{-\beta m}} = \sum_{k=0}^{\infty} p^k \prod_{r=0}^{k} \frac{1}{1 - q\mathrm{e}^{-\beta r}} \,, \tag{6.6}$$

This is indeed a true identity. It can be made manifest by writing the partition function in a way that is explicitly symmetric in p and q, as a sum over Young tableaux.

$$\sum_{N=0}^{\infty} q^N \prod_{m=0}^{N} \frac{1}{1 - p\mathrm{e}^{-\beta m}} = \sum_{N=0}^{\infty} q^N \sum_{n_0=0}^{\infty} \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} p^{\sum_m n_m} \mathrm{e}^{-\beta \sum_m m n_m} \tag{6.7}$$

If n_m is the number of rows of length $m+1$ in a tableau R (ignoring the first and longest, of length $l(R) = N+1$), the number of rows of the tableau is $h(R) = 1 + \sum_m n_m$ and the number of boxes in the tableau is $n(R) = N + 1 + \sum_m (m+1) n_m$. Then the sum can be written in the symmetric form:

$$\sum_{N=0}^{\infty} q^N \sum_{n_0=0}^{\infty} \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} p^{\sum_m n_m} \mathrm{e}^{-\beta \sum_m m n_m} = \sum_R q^{l(R)-1} p^{h(R)-1} \mathrm{e}^{-\beta(n(R) - l(R) - h(R) + 1)} \,. \tag{6.8}$$

6.4 Giant gravitons as subdeterminants

We end this subsection we describe, in more detail, the map between giant gravitons and operators. We have argued that single giant gravitons (states obtained from the quantization of linear polynomials) of energy k (i.e. the wavefunction $\psi = (w_{001})^k (w_{000})^{N-k}$) correspond to the state in the auxiliary counting problem of [23] that has k of the identical bosons in the first excited state of the harmonic oscillator, and all others in the ground state; this, in turn, corresponds to the gauge invariant function of the operator Z that is equal to $\sum_{\text{subsets}} z^{i_1} z^{i_2} \dots z^{i_k}$ (where $z^1, z^2 \dots z^N$ represent the various eigenvalues of Z, and the sum is over all subsets $\{i_1, i_2, \dots, i_k\}$ of $\{1, \dots, N\}$ with k elements). This, however, is precisely the character polynomial of $\mathrm{U}(N)$ in its k^{th} antisymmetric representation (representation with k boxes in the first column of a Young tableau). We have thus reproduced the standard map from giant gravitons to operators!

The considerations of the last paragraph are easily generalized. A basis for operators that correspond to m giant gravitons of arbitrary charge (polynomials of degree exactly m) is given by the character polynomials of the operator Z in representations R whose corresponding Young tableaux have boxes only in the first m columns [59, 60]. See [29] for a beautiful further generalization of this map to arbitrary $\frac{1}{8}$ BPS operators.

7. Homogeneous polynomials

In this section we determine the Hilbert Space that follows from the quantization of homogeneous polynomials of degree k. We denote this submanifold of phase space by M_k. The restriction to homogeneous polynomials of a given degree results in several simplifications as compared to the general case, the two most important of these are the facts that the surfaces $P(z) = 0$ always intersect the sphere for homogeneous P, so that the space of intersections of such surfaces with the unit sphere is a projective space with no holes, and also there are no identifications among the polynomials. While the symplectic form in the natural projective variables is not everywhere nonsingular, in this section we demonstrate that the singularities that exist are tame enough to control and that the holomorphic quantization of this space yields that is in precise accord with the conjecture at the beginning of this section. The rest of this section employs mathematical language and technology. We have attempted to make this section as self contained as possible.

The zero set of a homogeneous polynomial of any degree is invariant under the rescaling $z^i \to \lambda z^i$ for any complex number λ. The group S^1 acts freely on S^5 with $\mathbb{CP}^2$ as the quotient; the S^1 action corresponds to the Killing vector $(\mathrm{i} z^j \partial_{z^j} - \mathrm{i} \bar{z}^j \partial_{\bar{z}^j})$ on S^5. Therefore, the intersection of holomorphic homogeneous polynomials of degree k with the unit S^5 is a principal S^1-bundle over a degree k curve in $\mathbb{CP}^2$ (this means that S^1 acts freely transitively on the intersection). [18] The topological isomorphism classes of a principal S^1-bundle over a complex curve X are parameterized by $H^2(X, \mathbb{Z}) = \mathbb{Z}$. We note that the topological isomorphism class of the above S^1-bundle over a degree k curve is k.

[18] The genus of a smooth curve in $\mathbb{CP}^2$ of degree k is $\frac{(k-1)(k-2)}{2}$.

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As a consequence (2.6) induces a symplectic form on the space of degree k curves in $\mathbb{CP}^2$. The nature of this symplectic form is not difficult to determine. We first note that the D3-brane surfaces defined by homogeneous polynomials do not evolve in time (the time dependence in (2.1) may be absorbed into a time dependent overall rescaling). It follows as a consequence that $\omega_{\rm BI} = 0$ for such surfaces (this follows from (2.9), on noting that $e_\perp$ is tangent to these surfaces). Therefore, we have $\omega_{\rm full} = \omega_{\rm WZ}$.

The symplectic form $\omega_{\rm WZ}$ is constructed using the volume form on S^5. We will relate volume form ϵ_5 on S^5 with the volume form on $\mathbb{CP}^2$ given by the Fubini-Study metric.

We will denote by ω_2 the Fubini-Study Kähler form on $\mathbb{CP}^2$. Therefore, $\omega_2 \wedge \omega_2$ is a volume form on $\mathbb{CP}^2$; its total volume is 1. Let $f : S^5 \to \mathbb{CP}^2$ be the quotient map for the action of S^1. We will denote by $\mathrm{d}\theta$ the relative one form on S^5 for the projection f given by the form $\mathrm{d}\theta$ on S^1, where θ is the angle. (This is the Maurer-Cartan relative 1-form for the principal S^1-bundle.) Note that $f^*(\omega_2 \wedge \omega_2) \wedge \mathrm{d}\theta$ is a well-defined 5-form on S^5 (the wedge product is independent of the choice of the extension of the relative form $\mathrm{d}\theta$ to a differential 1-form on S^5). It is easy to see that

$$\epsilon_5 = \frac{\pi^2}{2} f^*(\omega_2 \wedge \omega_2) \wedge \mathrm{d}\theta \tag{7.1}$$

where ϵ_5 is the volume form of S^5.

Let V_k denote the complex vector space of homogeneous polynomials of degree k in three variables. The corresponding complex projective space will be denoted by $\mathbb{P}V_k$; so $\mathbb{P}V_k$ parameterizes all complex lines in the complex vector space V_k. Therefore, $\mathbb{P}V_k$ is biholomorphic to the complex projective space of complex dimension $(k+1)(k+2)/2 - 1$. We have a real submanifold

$$\mathcal{Y} \subset S^5 \times \mathbb{P}V_k \tag{7.2}$$

of real dimension $(k+1)(k+2)+1$ whose fibre over any point $p \in \mathbb{P}V_k$ is the intersection $S^5 \bigcap \{P(z^1, z^2, z^3) = 0\}$, where P is any homogeneous polynomial of degree k giving p (recall that points of $\mathbb{P}V_k$ parameterize lines in V_k). Let

$$\phi : \mathcal{Y} \to S^5 \tag{7.3}$$

be the projection to the first factor.

The 2-form $\omega_{\rm WZ}$ on $\mathbb{P}V_k$ is the integral(2.6)

$$\omega_{\rm WZ} = \frac{2N}{\pi^2} \int_{\mathcal{Y}/\mathbb{P}V_k} \phi^* \epsilon_5 \,, \tag{7.4}$$

where ϕ is the map in (7.3) and $\int_{\mathcal{Y}/\mathbb{P}V_k}$ is the integral of differential forms on $\mathcal{Y}$ along the fibres of the projection $\mathcal{Y} \to \mathbb{P}V_k$.

On the other hand, there is a complex submanifold

$$\mathcal{Z} \subset \mathbb{CP}^2 \times \mathbb{P}V_k \tag{7.5}$$

of complex dimension $(k+1)(k+2)/2$ whose fibre over any point $p \in \mathbb{P}V_k$ is the intersection $\mathbb{CP}^2 \bigcap \{P(z^1, z^2, z^3) = 0\}$, where P is any homogeneous polynomial of degree k giving p.

Therefore, $\mathcal{Y}$ (defined in (7.2)) is a principal S^1-bundle over $\mathcal{Z}$. Note that we have a commutative diagram of maps:

$$\begin{array}{ccc} \mathcal{Y} & \hookrightarrow & S^5 \times \mathbb{P}V_k \\ \downarrow & & \downarrow \\ \mathcal{Z} & \hookrightarrow & \mathbb{CP}^2 \times \mathbb{P}V_k \end{array}$$

where the projection $S^5 \times \mathbb{P}V_k \to \mathbb{CP}^2 \times \mathbb{P}V_k$ is $f \times \mathrm{Id}_{\mathbb{P}V_k}$ with f being the quotient by the action of S^1.

Let

$$\psi : \mathcal{Z} \to \mathbb{CP}^2 \tag{7.6}$$

be the projection. From (7.1) and (7.4) it follows that

$$\omega_{\mathrm{WZ}} = \frac{2N}{\pi^2} \int_{\mathcal{Y}/\mathbb{P}V_k} \phi^* \epsilon_5 = 2\pi N \int_{\mathcal{Z}/\mathbb{P}V_k} \psi^*(\omega_2 \wedge \omega_2)\,, \tag{7.7}$$

where ω_2 is the Fubini-Study Kähler form on $\mathbb{CP}^2$, and $\int_{\mathcal{Z}/\mathbb{P}V_k}$ is the integral of differential forms on $\mathcal{Z}$ along the fibres of the projection $\mathcal{Z} \to \mathbb{P}V_k$.

We will investigate ω_{WZ} using the identity in (7.7). Since $\omega_2 \wedge \omega_2$ is a $(2,2)$-form on $\mathbb{CP}^2$, as well as both ψ and the projection $\mathcal{Z} \to \mathbb{P}V_k$ are holomorphic maps, the fibre integral $\int_{\mathcal{Z}/\mathbb{P}V_k} \psi^*(\omega_2 \wedge \omega_2)$ is of type $(1,1)$. Since $\omega_2 \wedge \omega_2$ is closed we know that $\int_{\mathcal{Z}/\mathbb{P}V_k} \psi^*(\omega_2 \wedge \omega_2)$ is also closed. Although $\int_{\mathcal{Z}/\mathbb{P}V_k} \psi^*(\omega_2 \wedge \omega_2)$ may have some singularities, it defines a current on $\mathbb{P}V_k$ of degree two. This means that $\int_{\mathcal{Z}/\mathbb{P}V_k} \psi^*(\omega_2 \wedge \omega_2)$ is a continuous functional on the space of smooth differential forms on $\mathbb{P}V_k$ of degree $(k+1)(k+2)-4$; see chapter 1, section 2 of [47] for currents [19] Note that for any smooth differential form τ on $\mathbb{P}V_k$ of degree $(k+1)(k+2)-4$ we have

$$\int_{\mathbf{P}V_k} \tau \wedge \left(\int_{\mathcal{Z}/\mathbb{P}V_k} \psi^*(\omega_2 \wedge \omega_2) \right) = \int_{\mathcal{Z}} p_2^* \tau \wedge \psi^*(\omega_2 \wedge \omega_2)\,, \tag{7.9}$$

where p_2 is the projection of $\mathcal{Z}$ to $\mathbb{P}V_k$.

The cohomology class in $\mathcal{Z}$ defined by the form $\psi^*(\omega_2 \wedge \omega_2)$ lies in $H^2(\mathcal{Z}, \mathbb{Z})$. Therefore, $\frac{\omega_{\mathrm{WZ}}}{2\pi}$ is a closed current of type $(1,1)$, and it defines an element in $H^2(\mathbb{P}V_k, \mathbb{Z}) = \mathbb{Z}$. In other words, ω_{WZ} gives an integer, which is the proportionality constant of the cohomology class defined by $\frac{\omega_{\mathrm{WZ}}}{2\pi}$.

The above integer may be determined to be N using the arguments of section 3.6. Indeed, fixing a holomorphic homogeneous polynomial P_0 of degree $k-1$, consider the

[19] Note however that the form is not everywhere smooth. This is most easily seen from an explicit formula for the symplectic form. Let us work in the gauge $z^3 = 1$ on $\mathbb{CP}^2$. We find

$$\omega_{M_k} = \frac{N}{4\pi \mathrm{i}} \int_{C'} \left(\frac{\mathrm{d}\bar{z}^1 \wedge \mathrm{d}\bar{z}^2}{\mathrm{d}\overline{f}} \right) \wedge \left(\frac{\mathrm{d}z^1 \wedge \mathrm{d}z^2}{\mathrm{d}f} \right) \times \frac{1}{(1+|z^1|^2+|z^2|^2)^3} \frac{\mathrm{d}\overline{f} \wedge \mathrm{d}f}{2\mathrm{i}}\,, \tag{7.8}$$

where $\frac{\mathrm{d}z^1 \wedge \mathrm{d}z^2}{\mathrm{d}f}$ is defined to be equal to $\frac{\mathrm{d}z^1}{\frac{\partial f}{\partial z^2}} = -\frac{\mathrm{d}z^2}{\frac{\partial f}{\partial z^1}}$. While this is not manifest, it not difficult to verify that (7.8) is gauge invariant; i.e. (7.8) is invariant under any permutation of the indices $1, 2, 3$ of the projective coordinates of $\mathbb{CP}^2$. Note that the symplectic form has singularities at degenerations of the curve $P(z) = 0$, see appendix D.3 for an explicit example subspace of $\mathbb{P}V_k$ that corresponds to the homogeneous polynomials of the form $P_0 P$, where P runs over all nonzero homogeneous polynomials of degree one. This gives a map from $\mathbb{P}V_1 = \mathbb{CP}^2$ to $\mathbb{P}V_k$ of degree 1. From the expression of ω_{WZ} in (7.7) we know that the form ω_{WZ} on $\mathbb{P}V_k$ pulls back to the ω_{WZ} on $\mathbb{P}V_1 = \mathbb{CP}^2$. On the other hand, we know that the $\frac{\omega_{\mathrm{WZ}}}{2\pi}$ on $\mathbb{P}V_1 = \mathbb{CP}^2$ is N-times the positive generator of $H^2(\mathbb{CP}^2, \mathbb{Z}) = \mathbb{Z}$. Therefore, the cohomology class in $H^2(\mathbb{P}V_k, \mathbb{Z})$ given by $\frac{\omega_{\mathrm{WZ}}}{2\pi}$ is N-times the positive generator of $H^2(\mathbb{P}V_k, \mathbb{Z}) = \mathbb{Z}$.

Thus, the holomorphic line bundle $\mathcal{O}_{\mathbb{P}V_k}(N)$ is the unique holomorphic line bundle on $\mathbb{P}V_k$ whose first Chern class coincides with the cohomology class in $H^2(\mathbb{P}V_k, \mathbb{Z})$ given by $\frac{\omega_{\mathrm{WZ}}}{2\pi}$. Since ω_{WZ} is a current of type $(1,1)$, the holomorphic line bundle $\mathcal{O}_{\mathbb{P}V_k}(N)$ admits a Hermitian connection, whose (Hodge type) $(1,0)$-part may have singularities, such that curvature of the connection coincides with ω_{WZ}.

It thus follows that the holomorphic quantization of M_k is well defined. As the symplectic form is a closed $(1,1)$ the quantization may be performed holomorphically; the Hilbert space is simply the set of all degree N polynomials of the coefficients c_{n_1,n_1,n_3} (of homogeneous polynomials of degree k in three variables) with the usual implementation of U(3) generators on this space, in perfect agreement with (4.2).

8. Discussion

In this section we will comment on aspects of the procedure adopted in our paper, and discuss implications and generalizations or our results.

8.1 Gravitons from $D3$-branes

Our quantization of giant gravitons has reproduced the spectrum of all $\frac{1}{8}$ BPS states of Yang-Mills theory. Note that at strong coupling, some of these states are most naturally thought of as multiparticle superpositions of ordinary (rather than giant) gravitons. It follows that ordinary gravitons may be obtained from the quantization of small spherical $D3$-branes, in much the same way that they may be obtained from the quantization of strings.

Recall that the only properties of the symplectic form ω (and hence of the precise nature of the action on the world volume of the D3-brane) that we needed to derive (4.1) were U(3) invariance, smoothness properties and the cohomology class of ω. Any deformation of $\omega_{\mathrm{WZ}} + \omega_{\mathrm{BI}}$ that preserves these properties would give the same result for (4.1). It is certainly plausible that these three features are exact features of ω; this would explain why the crude considerations of this paper (based on a two derivative world volume action on the D3-brane) manage to reproduce the exact formula even for gravitons, (a feat that would seem to require considerably greater precision). Of course supersymmetry underlies this 'miracle', as it appears to permit us to truncate the daunting problem of the quantization of all solutions to the problem of quantizing a finite dimensional subspace with rigid topology.

8.2 Dual giant gravitons

It is well known that Mikhailov's solutions do not exhaust the list of $\frac{1}{8}$ D3-brane classical solutions. There exist well half BPS (and so in particular $\frac{1}{8}$ BPS) D3-brane puffed out into the AdS_5 rather than the S^5 directions.

It is familiar from the study of half BPS states that giant gravitons and dual giant gravitons are not independent solutions, but are instead dual to each other. We pause here to review this. The spectrum of half BPS states may be put in one to one correspondence with Young tableaux of SU(N). The state corresponding to a given Young tableaux may be regarded either as a collection of as many ordinary giant gravitons as the tableaux has columns or as a collection of a collection of as many dual giant gravitons as the Tableaux has rows. In particular a single dual giant graviton of angular momentum k may equally well be thought of as collection of k ordinary giant gravitons, each with unit angular momentum.

A similar equivalence seems to work for the full spectrum of $\frac{1}{8}$ BPS states. Recall that a single half BPS dual giant graviton has a world volume that wraps the S^3 (of a given radius) in AdS_5, but is located at a single point on S^5. Using the fact that $\frac{1}{8}$ BPS configurations are necessarily spherically symmetric on the S^3 in AdS_5, as they can carry no AdS angular momentum, the most general configuration of $\frac{1}{8}$ BPS giant gravitons is a simple superposition of (arbitrarily U(3) rotated) single dual giant gravitons. Moreover G. Mandal and N. Suryanarayana [33] have argued that the bosonic spectrum of single $\frac{1}{8}$ BPS giant graviton turns out to be that of a 3 dimensional bosonic harmonic oscillator. It follows that the spectrum of collection of N such dual gravitons reproduces the spectrum of ordinary giant gravitons derived in this paper.

Demanding the presence of exactly N dual giant gravitons is not as unreasonable as it may first seem [33]. Dual gravitons in the ground state of the Harmonic Oscillator carry no charge, and so are indistinguishable from nothing. Thus the restriction on number is really an upper bound, which is also rather intuitive, appearing to follow from the intuitive requirement that the flux at the center of AdS be positive.

Modulo cleaning up some loose ends, it thus seems that the partition function obtained by quantizing dual giant gravitons is identical to the partition function obtained by quantizing giant gravitons; further both of these are equal to the partition function over the classical chiral ring of [23]. Note that, according to this interpretation, the dual giant gravitons should be identified with the N bosons in the partition function (4.1).

8.3 A phase transition for the nucleation of a bubble?

The authors of [23] have argued that the partition function (4.1) undergoes a phase transition as a function of scaled chemical potentials, in the large N limit. Only a finite number (out of the N bosons) are out of their ground state in the 'low temperature' phase while only a finite number of the N bosons occupy the ground state in the 'high temperature' phase. From the viewpoint of (4.1), the phase transition between these phases is simply Bose condensation.

We will now investigate the bulk interpretation of these two phases. The partition function of the low temperature phase is identical to that of supersymmetric gravitons in $AdS_5 \times S^5$. It follows that the 'low temperature' phase should be identified with a gas of gravitons in an undeformed ambient $AdS_5 \times S^5$.

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On the other hand the 'high temperature' phase should be described by a bulk solution with vanishing 5 form flux at the origin of AdS_5 [20]; this is the bulk order parameter for the phase transition. It thus appears that the bulk dual to the high temperature phase should be the close analogue of an enhancon solution [67]. It would be fascinating to find the explicit bulk solution.

8.4 Extensions for the future

8.4.1 Generalization to IIB theory on $AdS_5 \times L^{\mathrm{abc}}$

Over the last few years a number of authors have discovered an infinite number of generalizations of the AdS/CFT conjecture [68 – 78]. These generalizations establish an equivalence between a class of $\mathcal{N}=1$ quiver gauge theory and IIB theory on $AdS_5 \times L^{\mathrm{abc}}$, where L^{abc} is a 5 dimensional space that may be thought of as the base of a (singular) six dimensional Calabi-Yau space.

It should be possible to evaluate the spectrum of giant gravitons (and dual giant gravitons) on $AdS_5 \times L^{\mathrm{abc}}$, and to compare this to the spectrum over the chiral ring of the corresponding $\mathcal{N}=1$ field theories. We hope to report on this in the near future.

8.4.2 Generalization to 1/16 BPS states

The spectrum of $\frac{1}{16}$ BPS states in $\mathcal{N}=4$ Yang Mills theory is much more intricate, and much more poorly understood, than its $\frac{1}{8}$ BPS counterpart. This problem may be attacked from three different directions

1. By an enumeration of the classical cohomology of the relevant supersymmetry operator (see (B.1)), a counting problem whose solution should yield the full spectrum of $\frac{1}{16}$ BPS states at least weak coupling.

2. By the construction and quantization of all $\frac{1}{16}$ BPS giant gravitons, a procedure that should yield the strong coupling supersymmetric spectrum of $\mathcal{N}=4$ Yang Mills theory at least at intermediate energies (energies larger than order unity but smaller than order N^2) and strong coupling.

3. By a study of the supersymmetric black holes of [40 – 44] and references therein (valid at strong coupling and energies larger than or of order N^2).

While the three different approaches listed above have logically distinct domains of validity, we suspect that all three approaches will give the same answer. In particular it is possible that the generalization of the quantization of one sixteenth BPS giant gravitons could reproduce the full (finite N finite λ) $\frac{1}{16}$ BPS spectrum. To this end we have already partially generalized Mikhailov's classical construction to one sixteenth giant gravitons. We plan to study this enlarged solution space and its quantization, and hope to have reportable results in the not too distant future.

[20] The origin may be defined as the fixed point of the SO(4) AdS_5 isometry.

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A. Geometric quantization

In this appendix, we give a brief review of the main ideas of geometric quantization, glossing over all subtleties. See, for instance, [79, 80] for more details.

A.1 The set up

Consider a manifold with a U(1) line bundle, whose curvature, ω is an invertible two form called the symplectic form. In local coordinates, $\omega = \mathrm{d}\theta = \omega_{ij}\frac{dx^i\wedge dx^j}{2}$ where ω_{ij} is an invertible matrix whose inverse we denote by ω^{ij}.

Given any function f on the manifold, we may be associate to it the vector field $(X_f)^i = \omega^{ij}\partial_j f$. The Poisson Bracket of two classical functions, f_1 and f_2 in phase space, is given by $\omega^{ij}\partial_i f_1 \partial_j f_2$.

We wish to quantize the classical description, replacing classical points on phase space with functions on phase space (states), and replacing real classical functions on phase space with Hermitian operators acting on states. We demand that this replacement is linear, maps the classical Poisson Bracket to the quantum commutator and maps the constant classical function to the constant quantum (multiplication) operator.

The correspondence $f \to -\mathrm{i}\hbar X_f$ is linear and maps the Poisson bracket to the commutator, but maps the constant function to the zero operator, and so is unacceptable. It is easy to check that the alternate map

$$f \to \omega^{ij}\partial_i f\,(\mathrm{i}\hbar\partial_j + \theta_j) + f \equiv (\mathrm{i}\hbar)\partial_i f \omega^{ij} D_j + f\,, \tag{A.1}$$

satisfies all our conditions, where the covariant derivative is

$$D_i = \partial_i - \frac{\mathrm{i}}{\hbar}\theta_i\,. \tag{A.2}$$

The covariant derivative is well defined only on functions that transform like charged fields under symplectic gauge transformation

$$\theta_i \to \theta_i + \partial_i u\,, \quad \phi \to \mathrm{e}^{\frac{\mathrm{i}u}{\hbar}}\phi\,. \tag{A.3}$$

Consequently, wavefunctions are sections of charge one under the symplectic line bundle, (A.2) is the connection and ω is its curvature.

The space of all such sections would form too large a Hilbert Space (yielding functions of both x and p). We should restrict attention to line bundles that are functions 'of p only'. One way to achieve this is to restrict attention to sections that are covariantly constant along an (arbitrarily chosen) 'polarization'. A polarization is defined as the collection of n dimensional vector spaces generated by locally defined n independent vector fields on the manifold, with two additional restrictions. The first restriction that the Lie bracket of any two of these vector fields, restricted to any point, belongs to the n dimensional subspace described above; in other words we have a foliation. The second restriction is that the point wise contraction of any two of these vectors with the symplectic form vanishes.

More pictorially, a choice of (real) polarization foliates phase space into an n parameter set of n dimensional Lagrangian submanifolds (a submanifold of phase space is Lagrangian if the restriction of the symplectic form is zero). Given a polarization, we are instructed to restrict attention to sections that obey

$$v^i D_i \phi = 0 \,, \tag{A.4}$$

where v^i is a tangent vector along the submanifolds. As the restriction of ω to these submanifolds is zero, this condition is integrable. The set of unit charge line bundles that obey (A.4) constitutes the Hilbert Space of our system.

One can always choose a symplectic potential that is 'adapted' to the polarization, i.e. its components along the Lagrangian submanifolds are zero. We will always make this choice in what follows.

Once we have adopted a choice of polarization, not all classical functions are associated with quantum operators. We require that the operator defined by (A.1) preserves the polarization condition (A.4). For any given physical application it is important that the polarization (A.4) be chosen such that all important operators pass this test.

We illustrate these ideas with a familiar example. Consider the quantum mechanics of a single particle in one dimension. Here $\omega = \mathrm{d}p \wedge \mathrm{d}x$. Choosing the polarization $D_x \phi = 0$, and choosing the symplectic potential $\theta = -x\,\mathrm{d}p$, we find that our Hilbert space consists of functions of p; the momentum operator maps to multiplication by p, while the position operator maps to $\mathrm{i}\hbar\partial_p$, in agreement with the usual formulae.

In order to complete the specification of our Hilbert space we must define an inner product on states. One may be tempted to use:

$$\langle \phi | \psi \rangle = \int \bar{\phi} \psi \frac{\omega^n}{n!(2\pi)^n} \,,$$

However, this would not map real observables to hermitian operators if θ were complex. The way around this is to replace $\bar{\phi}\psi$ with the hermitian structure (ϕ, ψ) of the bundle (i.e. a pointwise inner product for the charged fields that replaces their simple pointwise product) that is compatible with the connection. The last requirement translates into the equation

$$\partial_i (\phi, \psi) = (D_{\bar{i}} \phi, \psi) + (\phi, D_i \psi) \,.$$

For the connection (A.2), this means:

$$\begin{aligned}(\phi,\psi) &= \bar{\phi}\psi\, W(x)\,,\\ \partial_i W(x) &= \frac{\mathrm{i}}{\hbar}(\bar{\theta}_i-\theta_i)W(x)\,,\quad \overline{W}=W\,.\end{aligned} \tag{A.5}$$

It must also transform appropriately under gauge transformations. Note that (A.5) only needs to hold for those derivatives that do not appear in (A.4) and so W is not completely specified by (A.5). In the example considered above, we could choose W to be any unit normalized function of x, so the inner product reduces to the familiar $\int \mathrm{d}p\, \bar{\phi}(p)\psi(p)$.

While our description of the Hilbert space depends on a choice of polarization, under some circumstances Hilbert spaces formed with different polarizations may be shown to be unitarily related. See chapter 10 of [79] for details.

We have skipped over several important subtleties in this lightening review. Perhaps the most important of these is our omission of a discussion of the so called 'half form' metaplectic correction. We will not even describe this correction here, referring the interested reader, once again, to chapter 10 of [79].

A.2 Holomorphic quantization

We now turn to the 'coherent state' or holomorphic quantization of phase space. A complex structure on J phase space is said to be compatible with the symplectic form if ω is of type (1,1) with respect to J. The key simplification in this case is the ability to impose holomorphic polarization $D_{\bar{z}^i}\phi = 0$; the Hilbert Space may be identified with the space of square integrable holomorphic sections.

A.3 Kähler quantization

When ω is a nondegenerate $(1,1)$ form with respect to a complex structure J, and when ω obeys $\partial\omega = \bar{\partial}\omega = 0$, it may be thought of as a Kähler class, and may be derived locally from a Kähler Potential, $\omega = \mathrm{i}\partial\bar{\partial}K$.

If we use the Kähler form as a symplectic form, we can use the holomorphic polarization mentioned in the previous subsection. The symplectic potential $\theta = -\mathrm{i}\partial K$ is adapted to this polarization, equation (A.4) is solved by holomorphic sections and (A.5) is solved by $W = \exp\left(-\frac{K}{\hbar}\right)$.

A.4 Quantization of $\mathbb{CP}^{r-1}$

Consider the space $\mathbb{CP}^{r-1}$ built out of the r projective variables g^α, with a $(1,1)$ symplectic form whose cohomology class is that of $(2\pi N)\omega_{\mathrm{FS}}$. We will determine the Hilbert space that follows from the holomorphic quantization of this space. Note that the isomorphism class of a holomorphic line bundle on $\mathbb{CP}^{r-1}$ is determined by the first Chern class (as $\mathbb{CP}^{r-1}$ is simply connected and $H^2(\mathbb{CP}^{r-1},\mathcal{O}_{\mathbb{CP}^{r-1}}) = 0$). The first Chern class of the tautological line bundle $\mathcal{O}_{\mathbb{CP}^{r-1}}(1)$ on $\mathbb{CP}^{r-1}$ coincides with the cohomology class given by the Fubini-Study Kähler form. Suppose that we already know that the cohomology class of the symplectic form on $\mathbb{CP}^{r-1}$ under consideration is $2\pi N$-times the cohomology class given by the Fubini-Study Kähler form. The space of holomorphic sections of the N-th tensor power of the line bundle $\mathcal{O}_{\mathbb{CP}^{r-1}}(1)$ is the space of homogeneous polynomials of degree N with variables g^α for $\alpha = 1 \ldots r$. Therefore, the holomorphic quantization yields the Hilbert Space of homogeneous polynomials of degree N with variables g^α.

B. The SU(2|3) content of 1/8 cohomology

$\mathcal{N} = 4$ Yang-Mills has 6 scalar fields Φ_{ij}, 4 chiral fermions $\Psi_{i\alpha}$ and a gauge field $A_{\alpha\dot{\beta}}$, where lower SU(4) indices $i = 1 \ldots 4$ are antifundamental and upper indices are fundamental. The scalars obey the complex conjugation rules $\Phi^*_{ij} = \Phi^{ij}$ where $\Phi^{ij} = \frac{\epsilon^{ijkl}\Phi_{kl}}{2}$. The fermions are complex and their complex conjugates are $\overline{\Psi}^{i\dot{\alpha}}$.

The supersymmetry generators Q^i_α act on these fields as (factors and signs are schematic in the next two equations)

$$\begin{aligned}
[Q^i_\alpha, \Phi_{jk}] &= \delta^i_j \Psi_{k\alpha} - \delta^i_k \Psi_{j\alpha}\,, \\
\{Q^i_\alpha, \Psi_{j\beta}\} &= 2\mathrm{i}\, \delta^i_j f_{\alpha\beta} + \mathrm{i}\epsilon_{\alpha\beta}[\Phi_{jk}, \Phi^{ki}]\,, \\
\{Q^i_\alpha, \overline{\Psi}^j_{\dot{\beta}}\} &= -2\mathrm{i} D_{\alpha\dot{\beta}} \Phi^{ij}\,, \\
[Q^i_\alpha, A_{\beta\dot{\gamma}}] &= \epsilon_{\alpha\beta} \overline{\Psi}^i_{\dot{\gamma}}\,,
\end{aligned} \tag{B.1}$$

and the action of $\overline{Q}^{\dot{\alpha}}_i$ is obtained by taking complex conjugate.

Q^1_α and $\overline{Q}_{1\dot{\alpha}}$ generate a $\mathcal{N} = 1$ subalgebra of the $\mathcal{N} = 4$ algebra. We will now display the action of these supercharges on fields in $\mathcal{N} = 1$ language. Let $\Phi^{1\,m+1} = \bar{\phi}^m$ so that $\frac{1}{2}\epsilon_{p\,m\,n}\Phi^{m+1\,n+1} = \phi_p$. (here the indices m, n, p run from $1 \ldots 3$). Further let $\Psi_{1\alpha} = \lambda_\alpha$, and $\Psi_{m+1\,\alpha} = \psi_{m\alpha}$. Dropping the superscript on our special supercharges we have

$$\begin{aligned}
[Q_\alpha, \bar{\phi}^m] &= 0\,, \\
[Q_\alpha, \phi_m] &= \psi_{m\alpha}\,, \\
\{Q_\alpha, \psi_{m\beta}\} &= \mathrm{i}\epsilon_{\alpha\beta}\epsilon_{mnp}[\bar{\phi}^n, \bar{\phi}^p]\,, \\
\{Q_\alpha, \lambda_\beta\} &= 2\mathrm{i} f_{\alpha\beta} - \mathrm{i}\epsilon_{\alpha\beta}[\phi_m, \bar{\phi}^m]\,, \\
\{Q_\alpha, \bar{\psi}^m_{\dot{\beta}}\} &= -2\mathrm{i} D_{\alpha\dot{\beta}} \bar{\phi}^m\,, \\
\{Q_\alpha, \bar{\lambda}_{\dot{\beta}}\} &= 0\,, \\
[Q_\alpha, A_{\beta\dot{\gamma}}] &= \epsilon_{\alpha\beta}\bar{\lambda}_{\dot{\gamma}}\,.
\end{aligned} \tag{B.2}$$

From these equations we identify ψ and λ as the $\mathcal{N} = 1$ chiralino and gaugino respectively.

The one eighth BPS states we study in this paper are in one to one correspondence with states in the cohomology of the operators Q_α. It follows from (B.2) that all such operators are built out of simultaneously commuting 'letters' $\bar{\phi}^m$ and $\bar{\lambda}_{\dot{\alpha}}$ (see [23]). The resulting Hilbert space is in one one correspondence with the Fock space of N identical, noninteracting particles propagating in the potential of a 3 bosonic and 2 fermionic dimensional harmonic oscillator.

Notice that $\frac{1}{8}$ BPS states transform in representations of that part of the superconformal algebra, PSU(2,2|4), that commutes with the superalgebra generated by Q_α and their Hermitian conjugates. This commuting subalgebra is generated by $\overline{Q}_{m+1\,\dot{\alpha}} \equiv \widetilde{Q}_{m\,\dot{\alpha}}$ ($m = 1 \dots 3$) along with Hermitian conjugates, and is the compact superalgebra SU(2|3).

In Free Yang-Mills theory, and within the cohomology of Q_α, the letters $\bar{\phi}^m$ and $\bar{\lambda}_{\dot{\alpha}}$ transform in an irreducible representation of SU(2|3). In particular

$$\begin{aligned}
[\widetilde{Q}_{m\dot{\alpha}}, \bar{\phi}^n] &= \delta^n_m \bar{\lambda}_{\dot{\alpha}}\,, \\
\{\widetilde{Q}_{m\dot{\alpha}}, \bar{\lambda}_{\dot{\beta}}\} &= 0\,.
\end{aligned} \tag{B.3}$$

In the rest of this appendix we will study the unitary representations of this compact superalgebra. The only result derived below, that we use in the bulk of the paper, is easily stated. Let $M_{\frac{1}{8}}$ represent the $\frac{1}{8}$ BPS cohomology, and let $M^S_{\frac{1}{8}}$ represent that part of the cohomology that is formed only from the fields $\bar{\phi}^m$. We will demonstrate below that all states in $M_{\frac{1}{8}}$ may be obtained by the action of (an arbitrary number of applications of) $\widetilde{Q}^{\dot{\alpha}}$ on $M^S_{\frac{1}{8}}$. The reader who feels that (B.3) already makes this result quite plausible, and who is otherwise uninterested in the representation theory of superconformal algebrae, may skip to the next appendix.

B.1 PSU(2,2|4): Algebra and unitary constructions

In this brief subsection we recall the construction and representation theory of PSU(2,2|4); in order to appreciate how the SU(2|3) generators we use below are related to the more familiar symmetry generators of $\mathcal{N} = 4$ Yang-Mills.

The superalgebra $SU(4,2|2)$ has a simple unitary implementation on the Hilbert space of 4 bosonic and 4 fermionic oscillators. Let the bosonic oscillators be denoted by a_α, a^α, $b_{\dot{\beta}}$, $b^{\dot{\beta}}$ ($\alpha = 1, 2$, $\dot{\beta} = 1, 2$) (upper/lower indices are creation/annihilation operators). Let the fermionic oscillators be γ^i, γ_i ($i = 1 \dots 4$). The operators $Q^{i\alpha}$ are implemented by $a^\alpha \gamma^i$. Similarly $\overline{Q}^{\dot{\alpha}}_i = b^{\dot{\alpha}} \gamma_i$. Superconformal generators are given by the Hermitian conjugates of these formulae $S_{i\alpha} = a_\alpha \gamma_i$ and $\overline{S}^i_{\dot{\alpha}} = b_{\dot{\alpha}} \gamma^i$. Each of these operators commutes with the 'Supertrace'

$$ST = a^\alpha a_\alpha - b^{\dot{\beta}} b_{\dot{\beta}} - \gamma^i \gamma_i + 2\,.$$

As a consequence it is consistent to restrict attention to the sub Hilbert space $ST = 0$; we will do this in what follows.

Notice that

$$\begin{aligned}
\{S_{i\alpha}, Q^{j\beta}\} &= \{a_\alpha \gamma_i, a^\beta \gamma^j\} \\
&= a_\alpha a^\beta\, \delta^j_i - \delta^\beta_\alpha\, \gamma^j \gamma_i \\
&= J^\beta_\alpha\, \delta^j_i + \delta^\beta_\alpha\, R^j_i + \delta^j_i\, \delta^\beta_\alpha\, \frac{\Delta}{2}\,,
\end{aligned} \tag{B.4}$$

with

$$\begin{aligned}
\Delta &= \frac{a^\alpha a_\alpha + b^{\dot{\alpha}} b_{\dot{\alpha}} + 2}{2}\,, \\
J^\beta_\alpha &= \left(a_\alpha a^\beta - \delta^\beta_\alpha \frac{a_\gamma a^\gamma}{2} \right) = J^a (T^a)^\beta_\alpha\,, \\
R^j_i &= \left(\gamma_i \gamma^j - \delta^j_i \frac{\gamma_k \gamma^k}{4} \right) = R^p (\widetilde{T}^p)^j_i\,,
\end{aligned} \tag{B.5}$$

where T^a and $\widetilde{T}^p$, respectively, are the generators of SU(2) and SU(4) in the fundamental representation and we have used $ST = 0$ in the last line of (B.4). The operator Δ in (B.4) is the Hamiltonian or generator of scale transformations.

Irreducible lowest weight representations of PSU(2,2|4) contain a distinguished set of primary states that are all annihilated by $S_{\alpha i}$ and $\overline{S}^i_{\dot\beta}$. Such states have definite energy (eigenvalue of Δ) and appear in irreducible representations of $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)$.

Let R_k ($k = 1 \ldots 3$) denote the SU(4) Cartan generators whose k^{th} diagonal element is unity and $(k+1)^{\text{th}}$ diagonal element is minus one. We label representations of SU(4) by the eigenvalues of R_1, R_2, R_3 on the highest weight state in the representation. Equation (B.4) together with unitarity can be used to derive inequalities on the quantum numbers of primary states

$$
\begin{aligned}
&E \geq E_1\,,\\
&E \geq E_2\,,\\
&E_1 = 2j_1 + 2 - 2\delta_{j_1 0} + \frac{3R_1 + 2R_2 + R_3}{2}\,,\\
&E_2 = 2j_2 + 2 - 2\delta_{j_2 0} + \frac{3R_3 + 2R_2 + R_1}{2}\,.
\end{aligned}
\tag{B.6}
$$

B.2 Oscillator construction and unitarity of SU(2|3)

The superalgebra SU(2|3) is a subalgebra of PSU(2,2|4); this sub-superalgebra is represented on the subspace of the oscillator Hilbert space of the previous subsection defined by $a_\alpha|\psi\rangle = \gamma^1|\psi\rangle = 0$, i.e. states annihilated by $Q^{1\alpha} = a^\alpha\gamma^1$ and $S_{1\alpha} = a_\alpha\gamma_1$. On this subspace the constraint $ST = 0$ reduces to $ST' = -b^{\dot\beta} b_{\dot\beta} - \gamma^m\gamma_m + 1 = 0$. It is easy to check that

$$
\begin{aligned}
\{\widetilde{S}^m_{\dot\alpha}, \widetilde{Q}^{\dot\beta}_n\} &= b_{\dot\alpha} b^{\dot\beta}\delta^m_n - \delta^j_i \gamma_n\gamma^m\\
&= \overline{J}^{\dot\beta}_{\dot\alpha}\,\delta^m_n - \delta^{\dot\beta}_{\dot\alpha}\,U^m_n + \delta^m_n\,\delta^\beta_\alpha\,\frac{\Delta}{3}\,,
\end{aligned}
\tag{B.7}
$$

with

$$
\begin{aligned}
\Delta &= \frac{b^{\dot\alpha} b_{\dot\alpha} + 2}{2}\,,\\
\overline{J}^{\dot\beta}_{\dot\alpha} &= \left(b_{\dot\alpha} b^{\dot\beta} - \delta^{\dot\beta}_{\dot\alpha}\,\frac{b_{\dot\gamma} b^{\dot\gamma}}{2}\right) = \overline{J}^a (T^a)^{\dot\beta}_{\dot\alpha}\,,\\
U^m_n &= \left(\gamma_n\gamma^m - \frac{\gamma_p\gamma^p}{3}\,\delta^n_m\right) = U^b(\widetilde{T}^b)^m_n\,,
\end{aligned}
\tag{B.8}
$$

where T^a and $\widetilde{T}^b$ are generators of SU(2) and SU(3) in the fundamental representation.

Let U_m denote the diagonal SU(3) matrices with unity in the m^{th} row and negative unity in the $(m+1)^{\text{th}}$ row respectively. Note that when SU(2|3) is embedded within PSU(2,2|4) we have

$$
U_1 = R_2\,,\ U_2 = R_3\,. \tag{B.9}
$$

All unitary SU(2|3) representations may be obtained by acting on a set of distinguished primary or lowest weight states with an arbitrary number of $\widetilde{Q}_{m\dot\alpha}$ generators. Primary states are annihilated by all $\widetilde{S}^m_{\dot\alpha}$, and appear in representations of $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \Delta$ (here Δ is the energy operator). The requirement of unitarity, together with (B.7), yields inequalities on the energy of primary states of irreducible representations of SU(2|3) as a function of the $\mathrm{SU}(3) \times \mathrm{SU}(2)$ quantum numbers of these states. In particular we find

$$E \geq U_1 + 2U_2 + 3(j_2 + 1 - \delta_{j_2 0}) = E_2 + \frac{1}{2}(E_2 - E_1)\,, \tag{B.10}$$

where we have used (B.9) and (B.6).

B.3 Representation content of the 1/8 BPS cohomology

Recall that the $\frac{1}{8}$ BPS cohomology may be constructed via a two step procedure. We first construct a single particle Hilbert space $\mathcal{H}$, the Hilbert space of a three bosonic and two fermionic dimensional harmonic oscillator. We then second quantize this description, promoting states in the single particle Hilbert space to creation operators, and study the N particle spectrum of this second quantized formulation.

The first quantized Hilbert space is easily decomposed into representations of SU(2|3). Let M_p denote the (short) SU(2|3) representation with highest weights $(U_1, U_2) = (p, 0)$, $\overline{J} = 0$ and $\Delta = p$. Notice these quantum numbers saturate the bound (B.10); a circumstance that is partly responsible for their simplicity. It is not difficult to explicitly construct the representation M_p (for instance by using a p flavour version of the oscillator construction of the previous subsection). It turns out that M_p decomposes into representations of $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \Delta$ as

$$M_p = (p, 0; 0; p) + \left(p - 1, 0; \frac{1}{2}; p + \frac{1}{2}\right) + (p - 2, 0; 0; p + 1)\,, \tag{B.11}$$

where the numbers stand for $(U_1, U_2; \overline{J}; \Delta)$. It then follows that the single particle Hilbert space $\mathcal{H}$ of the previous paragraph decomposes into SU(2|3) representations as

$$\mathcal{H} = \sum_{p=0}^{\infty} M_p\,. \tag{B.12}$$

In more detail, let a_p denote the purely bosonic excitations in $\mathcal{H}$ with p quanta excited, let b_p denote states in $\mathcal{H}$ with p bosonic and one fermionic quanta occupied, and let c_p denote the states in $\mathcal{H}$ with p bosonic and two fermionic quanta occupied. The decomposition (B.11), in terms of states, is simply

$$M_p = a_p + b_{p-1} + c_{p-2}\,.$$

Note in particular that the purely bosonic a_p are the primary states (states in the representation $(p, 0; 0; p)$ that are annihilated by $\widetilde{S}^m_{\dot\alpha}$) in this decompositions; none of the other states (b_p or c_p) are annihilated by all $\widetilde{S}^m_{\dot\alpha}$.

Upon second quantizing the states a_p, b_p and c_p each become creation operators that inherit their SU(2|3) transformation properties from the corresponding states. The $\frac{1}{8}$ BPS cohomology is obtained by acting on a vacuum with N different creation operators (any of a_p, b_p, c_p) on the Fock vacuum. It follows that $\widetilde{S}^m_{\dot\alpha}$ annihilates states obtained by acting on the vacuum with a_p's only; further these are the only states annihilated by all $\widetilde{S}^m_{\dot\alpha}$. As a consequence the state obtained by second quantizing the subspace of $\mathcal{H}$ consisting of the the bosonic harmonic oscillator (with the fermionic oscillators in their vacuum) constitute all the lowest weight (primary) states under SU(2|3); all other states in the $\frac{1}{8}$ BPS cohomology may be obtained by acting on these primary states with (sufficient applications of) $\widetilde{Q}_{m\dot\alpha}$.

C. Details of the symplectic form and charges

In this appendix we present the algebraic manipulations that allow us to replace the symplectic form (2.4) with (2.5). We also present a couple of explicit formulae for the energy of giant gravitons as a function of the intersecting polynomial.

C.1 From the Wess-Zumino coupling

Let us write $\omega_{\rm WZ} = \frac{1}{2}\omega_{ij}\,\mathrm{d}c^i \wedge \mathrm{d}c^j$, where c^i parameterize solutions to the equations of motion. From (2.4), we have:

$$\begin{aligned}
\omega_{ij} &= \int \mathrm{d}^3\sigma\, \frac{\delta}{\delta c^i}\left[A_{\mu_0\mu_1\mu_2\mu_3}\frac{\partial x^{\mu_1}}{\partial\sigma^1}\frac{\partial x^{\mu_2}}{\partial\sigma^2}\frac{\partial x^{\mu_3}}{\partial\sigma^3}\right]\frac{\delta x^{\mu_0}}{\delta c^j} - i\leftrightarrow j \\
&= \int \mathrm{d}^3\sigma \left\{\partial_\nu A_{\mu_0\mu_1\mu_2\mu_3}\frac{\delta x^\nu}{\delta c^i}\frac{\delta x^{\mu_0}}{\delta c^j}\frac{\partial x^{\mu_1}}{\partial\sigma^1}\frac{\partial x^{\mu_2}}{\partial\sigma^2}\frac{\partial x^{\mu_3}}{\partial\sigma^3}\right. \\
&\qquad\left. + A_{\mu_0\mu_1\mu_2\mu_3}\frac{\delta x^{\mu_0}}{\delta c^j}\frac{\partial}{\partial\sigma^1}\left[\frac{\delta x^{\mu_1}}{\delta c^i}\right]\frac{\partial x^{\mu_2}}{\partial\sigma^2}\frac{\partial x^{\mu_3}}{\partial\sigma^3} + \text{cyclic}(1,2,3)\right\} - i\leftrightarrow j\,.
\end{aligned} \tag{C.1}$$

We deal with the second term by integrating by parts in σ^1 in the original version, but not the $i\leftrightarrow j$ version.

$$\begin{aligned}
\omega_{ij} = \int \mathrm{d}^3\sigma \Bigg\{ &\partial_\nu A_{\mu_0\mu_1\mu_2\mu_3}\frac{\delta x^\nu}{\delta c^i}\frac{\delta x^{\mu_0}}{\delta c^j}\frac{\partial x^{\mu_1}}{\partial\sigma^1}\frac{\partial x^{\mu_2}}{\partial\sigma^2}\frac{\partial x^{\mu_3}}{\partial\sigma^3} \\
&- \partial_\nu A_{\mu_0\mu_1\mu_2\mu_3}\frac{\delta x^\nu}{\delta c^j}\frac{\delta x^{\mu_0}}{\delta c^i}\frac{\partial x^{\mu_1}}{\partial\sigma^1}\frac{\partial x^{\mu_2}}{\partial\sigma^2}\frac{\partial x^{\mu_3}}{\partial\sigma^3} \\
&- \partial_\nu A_{\mu_0\mu_1\mu_2\mu_3}\frac{\partial x^\nu}{\partial\sigma^1}\frac{\delta x^{\mu_0}}{\delta c^j}\frac{\delta x^{\mu_1}}{\delta c^i}\frac{\partial x^{\mu_2}}{\partial\sigma^2}\frac{\partial x^{\mu_3}}{\partial\sigma^3} \\
&- A_{\mu_0\mu_1\mu_2\mu_3}\frac{\delta x^{\mu_0}}{\delta c^j}\frac{\delta x^{\mu_1}}{\delta c^i}\left[\frac{\partial^2 x^{\mu_2}}{\partial\sigma^1\partial\sigma^2}\frac{\partial x^{\mu_3}}{\partial\sigma^3} + \frac{\partial x^{\mu_2}}{\partial\sigma^2}\frac{\partial x^{\mu_3}}{\partial\sigma^1\partial\sigma^3}\right] \\
&- A_{\mu_0\mu_1\mu_2\mu_3}\frac{\partial}{\partial\sigma^1}\left[\frac{\delta x^{\mu_0}}{\delta c^j}\right]\frac{\delta x^{\mu_1}}{\delta c^i}\frac{\partial x^{\mu_2}}{\partial\sigma^2}\frac{\partial x^{\mu_3}}{\partial\sigma^3} \\
&- A_{\mu_0\mu_1\mu_2\mu_3}\frac{\delta x^{\mu_0}}{\delta c^i}\frac{\partial}{\partial\sigma^1}\left[\frac{\delta x^{\mu_1}}{\delta c^j}\right]\frac{\partial x^{\mu_2}}{\partial\sigma^2}\frac{\partial x^{\mu_3}}{\partial\sigma^3} + \text{cyclic}(1,2,3)\Bigg\}\,,
\end{aligned}$$

where the second and last terms come from the $i\leftrightarrow j$ version.

The last two terms combined consist of something symmetric in μ_0, μ_1 contracted with A, and therefore cancel. Regrouping terms and relabeling dummy indices:

$$\begin{aligned}
\omega_{ij} = &\int d^3\sigma \left(\frac{\delta x^\alpha}{\delta c^i}\frac{\delta x^\beta}{\delta c^j}\right)\left(\frac{\partial x^\gamma}{\partial \sigma^1}\frac{\partial x^\delta}{\partial \sigma^2}\frac{\partial x^\epsilon}{\partial \sigma^3}\right)(\partial_\alpha A_{\beta\gamma\delta\epsilon} + \text{cyclic}) \\
&- \left(\frac{\delta x^\alpha}{\delta c^i}\frac{\delta x^\beta}{\delta c^j}\right)\Bigg[\frac{\partial^2 x^\mu}{\partial\sigma^1\partial\sigma^2}\frac{\partial x^\nu}{\partial\sigma^3}(A_{\beta\alpha\mu\nu} + A_{\beta\mu\alpha\nu}) \\
&\qquad \frac{\partial^2 x^\mu}{\partial\sigma^2\partial\sigma^3}\frac{\partial x^\nu}{\partial\sigma^1}(A_{\beta\nu\alpha\mu} + A_{\beta\nu\mu\alpha}) \\
&\qquad \frac{\partial^2 x^\mu}{\partial\sigma^3\partial\sigma^1}\frac{\partial x^\nu}{\partial\sigma^2}(A_{\beta\alpha\nu\mu} + A_{\beta\mu\nu\alpha})\Bigg] \\
= &\int d^3\sigma \left(\frac{\delta x^\alpha}{\delta c^i}\frac{\delta x^\beta}{\delta c^j}\right)\left(\frac{\partial x^\gamma}{\partial \sigma^1}\frac{\partial x^\delta}{\partial \sigma^2}\frac{\partial x^\epsilon}{\partial \sigma^3}\right) F_{\alpha\beta\gamma\delta\epsilon}\,. \qquad \text{(C.2)}
\end{aligned}$$

As there are N units of flux through the sphere (setting the brane charge and the radius of S^5 to one), F is $2\pi N$, times the volume form of the S^5, divided by the volume of S^5. Therefore, the result is $\frac{2N}{\pi^2}$ times the volume swept out by the two deformations of the surface.

C.2 From the Born-Infeld term

We will denote the complex surface in $\mathbb{C}^3$ that appears in the $\frac{1}{8}$ BPS solutions of [28] by S and its intersection with S^5 (that is wrapped by the $D3$-brane) by Σ.

The Born-Infeld contribution to the symplectic form can be computed from its symplectic potential

$$\theta_{\rm BI} = \frac{N}{2\pi^2}\int_\Sigma d^3\sigma\,\sqrt{-g}\,g^{0\alpha}\frac{\partial x^\mu}{\partial\sigma^\alpha}\,G_{\mu\nu}\,\delta x^\nu\,. \qquad \text{(C.3)}$$

In a neighbourhood of a particular point of the $D3$-brane surface, we choose space-time coordinates $(t, x^1, x^2, u^1, u^2, u^3)$ such that the surface is given by x^1, x^2=constant and the contours u^i=constant are perpendicular to it. We fix gauge on the worldvolume by $\sigma^0 = t, \sigma^i = u^i$ so that x^i are the dynamical fields.

In these coordinates, δx^ν is a deformation perpendicular to the surface and the perpendicular velocity is

$$v_\perp^\mu = \frac{\partial x^\mu}{\partial t}\,.$$

We also have

$$g_{00} = -(1 - v_\perp^2)\,, \qquad g_{0i} = 0\,, \qquad g_{ij} = G_{ij} \equiv (g_s)_{ij}\,.$$

Therefore, $\theta_{\rm BI}$ can be written as

$$\theta_{\rm BI} = \frac{N}{2\pi^2}\int_\Sigma d^3\sigma\sqrt{g_s}\frac{v_\perp \cdot \delta x}{\sqrt{1 - v_\perp^2}}\,. \qquad \text{(C.4)}$$

In §6 of [28], Mikhailov showed that

$$\frac{\sqrt{g_s}d^3\sigma}{\sqrt{1 - v_\perp^2}} = 2\,\delta\big(|z^i|^2 - 1\big)\,d(\mathrm{vol}S)\ ,$$

where $\mathrm{d}(\mathrm{vol}S)$ is the induced volume form of the complex surface S. In addition, the velocity is the perpendicular component of $e_\parallel$, the unit vector in the direction of the generator of a simultaneous rotation in the three two-planes of $\mathbb{C}^3$. Therefore,

$$\theta_{\mathrm{BI}} = \frac{N}{\pi^2} \int_S \mathrm{d}(\mathrm{vol}C)\, \delta\big(|z^i|^2 - 1\big)\, (e_\parallel \cdot \delta x)\,. \tag{C.5}$$

Let J be the complex structure of $\mathbb{C}^3$. When we are not at the boundaries of phase space and Σ is fully three dimensional, the tangent space of Σ has a two dimensional subspace that is closed under J. Let e_ψ denote the unit vector in $T\Sigma$ that is perpendicular to this subspace.

The only tangent vector to S that is not perpendicular to δx is $J \cdot e_\psi$. However, it is perpendicular to $e_\parallel$:

$$(J \cdot e_\psi) \cdot e_\parallel = -e_\psi \cdot (J \cdot e_\parallel) = e_\psi \cdot e_\perp\,,$$

where $e_\perp$ is the unit vector perpendicular to S^5 in $\mathbb{C}^3$, which is manifestly perpendicular to e_ψ.

Let n_1 and n_2 be orthogonal unit vectors perpendicular to S, such that $J \cdot n_1 = n_2$ and $J \cdot n_2 = -n_1$. We have

$$e_\parallel \cdot \delta x = (e_\parallel \cdot n_1)(\delta x \cdot n_1) + (e_\parallel \cdot n_2)(\delta x \cdot n_2) = (e_\perp \cdot n_1)(\delta x \cdot n_2) - (e_\perp \cdot n_2)(\delta x \cdot n_1)\,.$$

Parameterizing the surface S with coordinates $\sigma^1 \ldots \sigma^4$, we can rewrite (C.5) as:

$$\begin{aligned} \theta_{\mathrm{BI}} &= \frac{N}{\pi^2} \int_S \mathrm{d}^4\sigma\, \epsilon_{\mu_1 \cdots \mu_6} \left[\frac{\partial x^{\mu_1}}{\partial \sigma^1} \cdots \frac{\partial x^{\mu_4}}{\partial \sigma^4}\right] n_1^{\mu_5} n_2^{\mu_6}\, 2(n_1)_{[\alpha}(n_2)_{\beta]} \left(e_\perp^\alpha \delta x^\beta\right)\, \delta\big(|z^i|^2 - 1\big) \\ &= \frac{N}{\pi^2} \int_S \mathrm{d}^4\sigma\, \epsilon_{\mu_1 \cdots \mu_6} \left[\frac{\partial x^{\mu_1}}{\partial \sigma^1} \cdots \frac{\partial x^{\mu_4}}{\partial \sigma^4}\right] e_\perp^{\mu_5} \delta x^{\mu_6}\, \delta\big(|z^i|^2 - 1\big)\,. \end{aligned} \tag{C.6}$$

This is can be interpreted geometrically as follows:

- Compute $\frac{N}{2\pi^2}$ times the volume swept out by the surface S under deformations by $e_\perp$ and δx inside a sphere of radius r;
- differentiate with respect to r;
- set $r = 1$.

C.3 Explicit formulae for the energy

The energy $E = \sum_m L^m$ is of particular interest, and may be determined to be [28]

$$E = \frac{2N}{\omega_3} \int_S \mathrm{d}(\mathrm{vol}S)\, \delta\!\left(\sum_i |z^i|^2 - 1\right), \tag{C.7}$$

where S is the 4 real dimensional surface of the curve $P(z^i) = 0$ and $\mathrm{d}(\mathrm{vol}S)$ is the pull back of the volume form on $\mathbb{R}^6$ onto S and ω_3 is the volume of the unit 3 sphere. This formula may be rewritten as

$$E = \frac{2N}{\omega_3} \int \mathrm{d}^3 z\, \mathrm{d}^3 \bar{z}\, |\partial_i P|^2 \delta(P) \delta(\bar{P}) \delta\!\left(\sum_i |z^i|^2 - 1\right). \tag{C.8}$$

D. Curves of degree two

The family of holomorphic curves

$$c_{ij}z^iz^j = 1\,, \tag{D.1}$$

displays more or less all the features that complicate the quantization of holomorphic curves. As a consequence, a complete and explicit analysis of the quantization of these curves could be very valuable. We have not yet performed this (algebraically complicated) analysis. In this appendix we lay the ground for this analysis by laying out the issues and setting up the problem.

D.1 U(3) action

The matrix

$$M_i^{\ k} = c_{ij}\bar{c}^{jk} \tag{D.2}$$

is hermitian and can be diagonalized by a U(3) transformation: $M=\mathrm{diag}(\lambda_1^2,\lambda_2^2,\lambda_3^2)$. One can show that the most general symmetric matrix satisfying (D.2) is[21]

$$c = \mathrm{diag}(\lambda_1 e^{i\alpha_1}, \lambda_2 e^{i\alpha_2}\lambda_3 e^{i\alpha_3})\,.$$

We can then use the remaining $\mathrm{U}(1)^3$ symmetry to set the phases to zero. We conclude that any matrix c_{ij} may be written in the form

$$C = UDU^T\,,$$

where U is a U(3) matrix and D is a diagonal matrix with real positive eigenvalues. Notice that, in this way of writing it, the 12 real degrees of freedom of a complex symmetric matrix have been distributed into the 9 degrees of freedom of U and the 3 degrees of freedom of D.

D.2 Holes

The nearest the curve $\sum_i \lambda_i(x^i)^2 = 1$ approaches the origin is the minimum of $\frac{1}{\sqrt{\lambda_i}}$ for $i = 1\ldots 3$. When $\lambda_i < 1$ for $i = 1\ldots 3$ the curve fails to intersect the unit sphere; this region of non-intersection represents a unit cube in the positive octant of $\lambda_1,\lambda_2,\lambda_3$ space. The complement of this cube in the positive octant is in one to one correspondence with the intersections $\lambda_i(x^i)^2 = 1$ with the unit 5 sphere.

We now study the boundary of this region — the faces, edges and vertex of this cube — in more detail. Consider the face at $\lambda_1 = 1$ with $\lambda_2,\lambda_3 < 1$. Such a curve intersects the unit 5 sphere at exactly two points $x = \pm 1$, $y = z = 0$. This boundary is an 11 dimensional manifold in the 12 dimensional space of $c_{ij}s$; the 11 dimensions being spanned by λ_2,λ_3 and an arbitrary U(3) matrix.

Now let us turn to the edges of the cube, for example $\lambda_1 = \lambda_2 = 1$. The curve intersects the unit 5 sphere on such an edge along the circle $x^2 + y^2 = 1$ with x, y real. These edges

[21] This is an oversimplification. When two of the eigenvalues are equal there is a more general c. This corresponds to the U(2) subgroup of U(3) that is not fixed by diagonalizing M. A similar remark applies to the situation where all three eigenvalues are equal.

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constitute a 9 dimensional subspace in the space of coefficients c_{ij}; the 9 dimensions are spanned by λ_3 together with the elements $\mathrm{U}(3)/\,\mathrm{SO}(2)$ (note that an SO(2) subgroup of U(3) acts in a manner so as to leave a curve with $\lambda_1 = \lambda_2$ invariant).

Finally the corner of the cube has $\lambda_1 = \lambda_2 = \lambda_3 = 1$. This curve intersects the unit 5 sphere on the 2 sphere $x^2 + y^2 + z^2 = 1$ with all of x, y, z real. Such curves form a 6 dimensional subspace in the space of coefficients, the six dimensions parameterize $\mathrm{U}(3)/\,\mathrm{SO}(3)$.

It would be fascinating to study ω_{WZ} and ω_{full} on the c_{ij} space, and in particular to investigate whether the structure of these forms permits one to shrink away the cubic hole described in this section in a smooth manner, as our conjecture suggests should be the case. We will not address this problem in this paper.

D.3 Degenerations

In order to study a second kind of potential singularity of the symplectic form, let us study the limit $c_{ij} \to \infty$ for all i, j. In this limit our curve reduces effectively to a generic homogeneous degree two polynomial, which may, up to U(3) rotations, be chosen to have the form

$$x^2 + \lambda y^2 + \epsilon z^2 = 0\,, \quad 0 \le \lambda, \epsilon \le 1\,. \tag{D.3}$$

(D.3) degenerates into two curves when the l.h.s. of (D.1) factorizes, i.e. when $\det c = 0$. Note that $\epsilon = 0$ is the degeneration we will concentrate on. It is interesting to study the symplectic form in the neighbourhood of this degeneration point.

We use (7.8):

$$\omega_{M_k} = \frac{N}{4\pi \mathrm{i}} \int_{C'} \left(\frac{\mathrm{d}\bar{z}^1 \wedge \mathrm{d}\bar{z}^2}{\mathrm{d}\overline{f}} \right) \wedge \left(\frac{\mathrm{d}z^1 \wedge \mathrm{d}z^2}{\mathrm{d}f} \right) \times \frac{1}{(1 + |z^1|^2 + |z^2|^2)^3} \frac{\mathrm{d}\overline{f} \wedge \mathrm{d}f}{2\mathrm{i}}\,,$$

It is helpful to make the variable change

$$\alpha = x + \mathrm{i}\sqrt{\lambda} y\,, \qquad \beta = x - \mathrm{i}\sqrt{\lambda} y\,.$$

Working in the gauge $z = 1$, we can solve (D.3) by $\beta = -\epsilon/\alpha$. We have:

$$\begin{aligned}
\frac{\mathrm{d}z^1 \wedge \mathrm{d}z^2}{\mathrm{d}f} &\to -\frac{\mathrm{d}\alpha}{2\mathrm{i}\sqrt{\lambda}\frac{\partial f}{\partial \beta}} = -\frac{\mathrm{d}\alpha}{2\mathrm{i}\sqrt{\lambda}\alpha}\,, \\
x &= \frac{\alpha + \beta}{2} = \frac{1}{2}\left(\alpha - \frac{\epsilon}{\alpha}\right), \\
y &= \frac{\alpha - \beta}{2\mathrm{i}\sqrt{\lambda}} = \frac{1}{2\mathrm{i}\sqrt{\lambda}}\left(\alpha + \frac{\epsilon}{\alpha}\right), \\
|x|^2 + |y|^2 + |z|^2 &= 1 + \frac{1}{4}\left\{ \left(\frac{1}{\lambda} + 1\right)\left(|\alpha|^2 + \frac{\epsilon^2}{|\alpha|^2}\right) + \left(\frac{1}{\lambda} - 1\right)\epsilon\left(\frac{\alpha}{\bar{\alpha}} + \frac{\bar{\alpha}}{\alpha}\right)\right\}.
\end{aligned}$$

(7.8) becomes

$$\omega = \frac{N}{16\pi \mathrm{i}\lambda} \int \frac{\mathrm{d}\bar{\alpha}\mathrm{d}\alpha}{|\alpha|^2} \frac{z^i z^i \bar{z}_k \bar{z}_l}{\left[1 + \frac{1}{4}\left\{\left(\frac{1}{\lambda} + 1\right)\left(|\alpha|^2 + \frac{\epsilon^2}{|\alpha|^2}\right) + \left(\frac{1}{\lambda} - 1\right)\epsilon\left(\frac{\alpha}{\bar{\alpha}} + \frac{\bar{\alpha}}{\alpha}\right)\right\}\right]^3} \frac{\mathrm{d}\bar{c}^{kl} \wedge \mathrm{d}c_{ij}}{2\mathrm{i}}\,.$$

So the most general integral we need to evaluate is:

$$\frac{N}{8\pi\lambda}\int\frac{\mathrm{d}\alpha\mathrm{d}\bar{\alpha}}{2\mathrm{i}|\alpha|^6}\frac{\alpha^m\bar{\alpha}^n}{\left[1+\frac{1}{4}\left\{\left(\frac{1}{\lambda}+1\right)\left(|\alpha|^2+\frac{\epsilon^2}{|\alpha|^2}\right)+\left(\frac{1}{\lambda}-1\right)\epsilon\left(\frac{\alpha}{\bar{\alpha}}+\frac{\bar{\alpha}}{\alpha}\right)\right\}\right]^3}$$
$$=\frac{N}{8\pi\lambda}\int\mathrm{d}r\mathrm{d}\phi\,\frac{r^{m+n+1}\mathrm{e}^{\mathrm{i}(m-n)\phi}}{\left[r^2+\frac{1}{4}\left\{\left(\frac{1}{\lambda}+1\right)(r^4+\epsilon^2)+\left(\frac{1}{\lambda}-1\right)2\epsilon r^2\cos 2\phi\right\}\right]^3}\,,$$

with $m,n=0,1,2,3,4$.

This becomes considerably easier at the special point $\lambda=1$:

$$N\int\mathrm{d}x\frac{x^m\,\delta_{mn}}{(x^2+2x+\epsilon^2)^3}\,.$$

This gives the following beautiful formulae:

$$\begin{aligned}
\omega_{\overline{11},11}=\omega_{\overline{22},22}&=\frac{N}{256\mathrm{i}}\left(\frac{2(2-11\epsilon^2)\sqrt{1-\epsilon^2}-\epsilon^2(4+5\epsilon^2)\log\left[\frac{2-\epsilon^2-2\sqrt{1-\epsilon^2}}{\epsilon^2}\right]}{(1-\epsilon^2)^{5/2}}\right),\\
\omega_{\overline{22},11}&=-\frac{N}{256\mathrm{i}}\left(\frac{2(2+\epsilon^2)\sqrt{1-\epsilon^2}-\epsilon^2(4-\epsilon^2)\log\left[\frac{2-\epsilon^2-2\sqrt{1-\epsilon^2}}{\epsilon^2}\right]}{(1-\epsilon^2)^{5/2}}\right),\\
\omega_{\overline{12},12}&=\frac{N}{256\mathrm{i}}\left(\frac{2(2-5\epsilon^2)\sqrt{1-\epsilon^2}-3\epsilon^4\log\left[\frac{2-\epsilon^2-2\sqrt{1-\epsilon^2}}{\epsilon^2}\right]}{(1-\epsilon^2)^{5/2}}\right),\\
\omega_{\overline{13},13}=\omega_{\overline{23},23}&=\frac{N}{64\mathrm{i}}\left(\frac{2(1+2\epsilon^2)\sqrt{1-\epsilon^2}+3\epsilon^2\log\left[\frac{2-\epsilon^2-2\sqrt{1-\epsilon^2}}{\epsilon^2}\right]}{(1-\epsilon^2)^{5/2}}\right),\\
\omega_{\overline{33},11}=\omega_{\overline{33},22}&=\frac{N}{64\mathrm{i}}\left(\frac{6\epsilon\sqrt{1-\epsilon^2}+\epsilon(2+\epsilon^2)\log\left[\frac{2-\epsilon^2-2\sqrt{1-\epsilon^2}}{\epsilon^2}\right]}{(1-\epsilon^2)^{5/2}}\right),\\
\omega_{\overline{33},33}&=-\frac{N}{32\mathrm{i}}\left(\frac{6\sqrt{1-\epsilon^2}+(2+\epsilon^2)\log\left[\frac{2-\epsilon^2-2\sqrt{1-\epsilon^2}}{\epsilon^2}\right]}{(1-\epsilon^2)^{5/2}}\right).
\end{aligned}\tag{D.4}$$

All other components can either be determined from symmetry and reality or are zero. The component $\omega_{\overline{33},33}$ has a $\log\epsilon$ singularity at $\epsilon=0$. The other non-zero components have non-analytic $\epsilon^n\log\epsilon$ behaviour, but are finite. These are mild singularities that can be integrated over.

E. Holomorphic surfaces that touch the unit sphere

In this subsection we will demonstrate that ω_{WZ} vanishes at every point on the boundary of H.

Consider any polynomial $P(z)$ on the boundary of H. It follows that the surface $P(z)=0$ intersects the unit sphere on some submanifold M', but that there exist infinitesimal deformations of $P(z)$ under which this intersection goes to zero. In this subsection we will demonstrate that the submanifold M' cannot have dimension 3 (i.e. is at most 2 dimensional). As $\omega_{\rm WZ}$ is obtained by integrating a well behaved 3 form over the intersection M', it then follows that $\omega_{\rm WZ}$ vanishes at the boundaries of H.

Consider $\mathbb{R}^6$ equipped with the standard inner product structure defined by

$$\|(x_1,\ldots,x_6)\|^2 = \sum_{i=1}^{6} x_i^2 .$$

Let $S^5 = \{(x_1,\ldots,x_6) \mid \sum_{i=1}^{6} x_i^2 = 1\}$ be the five-dimensional sphere.

Identify $\mathbb{R}^6$ with $\mathbb{C}^3$ by sending $(x_1,\ldots,x_6)$ to $(x_1+\mathrm{i}x_2\,, x_3+\mathrm{i}x_4\,, x_5+\mathrm{i}x_6)$. Using this identification, the operation of multiplication by i on $\mathbb{C}^3$ gives a linear operator J on $\mathbb{R}^6$. We have $J^2 = -\mathrm{Id}$, and J is orthogonal with respect to the standard inner product.

Let N denote the section of the normal bundle of $S^5 \subset \mathbb{R}^6$ that sends any $\underline{x} \in S^5$ to the element $\underline{x} \in T_{\underline{x}}\mathbb{R}^6 = \mathbb{R}^6$. Therefore, N is the unit normal vector field that points outwards.

Since J is orthogonal, it follows that $J(N)$ is a vector field on the manifold S^5. Let ω denote the C^∞ one-form on S^5 defined as follows: for any point $\underline{x} \in S^5$ and any tangent vector $v \in T_{\underline{x}}S^5$,

$$\omega(v) := \langle v\,, J(N(\underline{x}))\rangle . \tag{E.1}$$

Let

$$\mathcal{F} := \mathrm{kernel}(\omega) \subset TS^5$$

be the distribution on S^5. So, $\mathcal{F}$ is a C^∞ subbundle of TS^5 whose fibre over any $\underline{x} \in S^5$ is $J(N(\underline{x}))^\perp \bigcap T_{\underline{x}}S^5$.

We will show that for any point $\underline{x} \in S^5$, the subspace $\mathcal{F}_{\underline{x}} \subset T_{\underline{x}}\mathbb{R}^6 = \mathbb{R}^6$ is closed under the operator J. Take any $v \in \mathcal{F}_{\underline{x}}$. Since J is orthogonal,

$$\langle J(v)\,, N(\underline{x})\rangle = \langle J^2(v)\,, J(N(\underline{x}))\rangle = -\langle v\,, J(N(\underline{x}))\rangle = 0\,,$$

and $\langle J(v)\,, J(N(\underline{x}))\rangle = \langle v\,, N(\underline{x})\rangle = 0$. Therefore, $J(v) \in \mathcal{F}_{\underline{x}}$, hence the subspace $\mathcal{F}_{\underline{x}}$ is closed under J.

Consider the two-form $\mathrm{d}\omega$ on S^5, where ω is defined in (E.1).

Claim. For any point $\underline{x} \in S^5$, the restriction of $\mathrm{d}\omega$ to $\mathcal{F}_{\underline{x}}$ is a symplectic form.

To prove the claim, consider U(3) as a subgroup of SO(6) using the identification of $\mathbb{R}^6$ with $\mathbb{C}^3$. This subgroup commutes with J. Since ω is invariant under the action of U(3), and U(3) acts transitively on S^5, it is enough to check the above claim for one point of S^5. Take $\underline{y} = (0,0,0,0,0,1)$. Then $J(N(\underline{y})) = (0,0,0,0,-1,0)$, and

$$\mathcal{F}_{\underline{y}} = \mathbb{R}^4 := (\mathbb{R},\mathbb{R},\mathbb{R},\mathbb{R},0,0) \subset \mathbb{R}^6 = T_{\underline{y}}\mathbb{R}^6 . \tag{E.2}$$

Consider the one-form $\widetilde{\omega}$ on $\mathbb{R}^6$ that sends any tangent vector $v \in T_{\underline{x}}$ to $\langle v\,, \underline{x}\rangle$. Therefore, ω is the restriction of $\widetilde{\omega}$ to S^5. The restriction of $\mathrm{d}\widetilde{\omega}$ to the subspace $\mathcal{F}_{\underline{y}} = \mathbb{R}^4$ in (E.2) coincides with the symplectic form $2(\mathrm{d}x_1 \bigwedge \mathrm{d}x_2 + \mathrm{d}x_3 \bigwedge \mathrm{d}x_4)$ on $\mathbb{R}^4$.

Therefore, for any $\underline{x} \in S^5$, the restriction of $\mathrm{d}\omega$ to $\mathcal{F}_{\underline{x}}$ is a symplectic form. Hence the restriction of $\mathrm{d}\omega \bigwedge \mathrm{d}\omega$ to $\mathcal{F}_{\underline{x}}$ is nonzero. Since $\mathcal{F}$ is, by definition, the kernel of ω, this implies that

$$\theta := \omega \wedge \mathrm{d}\omega \wedge \mathrm{d}\omega \tag{E.3}$$

is a nowhere vanishing top form on S^5.

Let M be a submanifold of S^5 such that for each point $m \in M$ we have

$$T_m M \subset \mathcal{F}_m \subset T_m S^5 \,. \tag{E.4}$$

We will show that

$$\dim M \leq 2 \,.$$

To prove this by contradiction, assume that $\dim M = 3$. Fix a point $m_0 \in M$. Let $(t_1, t_2, t_3, t_4, t_5)$, $-\epsilon < t_i < \epsilon$, be smooth coordinates on S^5 defined on an open subset U containing m_0 such that

$$M \cap U = \{(t_1, t_2, t_3, t_4, t_5) \mid t_4 = 0 = t_5\} \,.$$

Let $\omega|_U = \sum_{i=1}^5 f_i \mathrm{d}t_i$, where ω is the form defined in (E.1), and f_i are some smooth functions on U. From (E.4) and the definition of ω it follows that f_1, f_2 and f_3 vanish on $M \bigcap U$.

Since

$$(\mathrm{d}\omega)|_U = \sum_{i=1}^5 \sum_{j=1}^5 \frac{\partial f_i}{\partial t_j} \mathrm{d}t_j \wedge \mathrm{d}t_i \,,$$

and f_k, $1 \leq k \leq 3$, vanish on $M \bigcap U$, at any point $m \in M \bigcap U$,

$$(\mathrm{d}\omega)(m) = \sum_{i=1}^3 \sum_{j=4}^5 \frac{\partial f_i}{\partial t_j}(m)\, \mathrm{d}t_j \wedge \mathrm{d}t_i + \sum_{i=4}^5 \sum_{j=1}^5 \frac{\partial f_i}{\partial t_j}(m)\, \mathrm{d}t_j \wedge \mathrm{d}t_i \,.$$

From this it follows that the form θ defined in (E.3) vanishes at m. Indeed, both $\omega(m)$ and $\mathrm{d}\omega(m)$ are contained in the ideal generated by $\mathrm{d}t_4$ and $\mathrm{d}t_5$ (each term in their expression contains either $\mathrm{d}t_4$ or $\mathrm{d}t_5$). Hence

$$\theta(m) = \omega(m) \wedge (\mathrm{d}\omega)(m) \wedge (\mathrm{d}\omega)(m) = 0 \,.$$

This contradicts the fact that the five-form θ is nowhere vanishing. Hence $\dim M \leq 2$.

If Z is the zero set of a complex polynomial in three variables such that it can be made disjoint from S^5 by arbitrarily small perturbations, then for any $x \in Z \bigcap S^5$ the intersection of Z and S^5 is not transversal. This means that the inclusion

$$T_x Z \subset \mathcal{F}_x$$

holds. Hence by the above, $\dim_{\mathbb{R}} Z \bigcap S^5 < 3$.

E.1 $\theta_{\rm BI}$ on the boundary

In section 3.2 we have argued that $\theta_{\rm BI}$ vanishes on the boundary of solution space in a distributional sense. Here we study the restriction of $\theta_{\rm BI}$ to the boundary in more detail; in particular we will see that it is zero only as a current and not pointwise.

The generic surface $P(z) = 0$ cuts the unit five sphere (at a 'nonzero angle' on a 3 surface; the the volume δV of such a surface contained within a δr shell of the unit sphere is, consequently, of order δr. Consequently contraction of $\theta_{\rm BI}$ with an arbitrary vector is generically finite and nonzero. Now let us turn to surfaces $P(z) = 0$ where $\rho(P(z)) = 1$. Such surfaces are different from generic in two important ways. First they touch the unit 5-sphere on a p dimensional surface with $p \leq 2$. Second they touch the unit 5 sphere at 'zero angle' (rather than cutting it at a finite angle); as a consequence the 4 volume contained within the shell of thickness δr around the unit sphere is of order $(\delta r)^{\frac{4-p}{2}}$.

It follows that the restriction of $\theta_{\rm BI}$ to the boundary of the hole vanishes at those boundary points where the intersection is zero or one dimensional. We also note that the boundary surfaces whose intersection with the unit sphere is 2 dimensional form low dimensional submanifolds on the space of all boundary polynomials (see appendix D for an example). As a consequence $\omega_{\rm BI}$ vanishes as a current on the boundary, which means that its integral against any genuine form vanishes.

F. Symplectic form for linear functions

F.1 Coordinates and parametrization

In this appendix we study the symplectic form for linear polynomials:

$$c_i z^i - 1 = 0\,.$$

U(3) rotations may be used to rotate this Polynomial into the curve

$$c_0 z - 1 = 0\,, \tag{F.1}$$

where $|c_0|^2 = |a|^2 + |b|^2 + |c|^2$. The intersection of (F.1) with the unit 5 sphere is a three sphere of squared radius $1 - 1/|c_0|^2$. Note that (F.1) fails to intersect the unit sphere for $|c_0|^2 < 1$. Using, for instance (C.7) we find that the energy of this giant graviton is

$$E = N\left(1 - \frac{1}{|c_0|^2}\right). \tag{F.2}$$

A formula that, of course, is valid only for $|c_0| > 1$.

We will now determine the symplectic form on the space of linear polynomials. Using U(3) invariance it will be sufficient to consider the neighbourhood of the special curve (F.1). Let σ^i $(i = 1 \ldots 3)$ represent any three coordinates on a unit S^3 embedded in $\mathbb{C}^2$ (we will never need to specify what these are) such that $x = x^0(\sigma^i)$ and $y = y^0(\sigma^i)$. We will choose to parameterize points on the target space S^5 (this is our choice of target space variables

x^μ in (2.8)) by z together with the three coordinates σ^i, in terms of which the embedding $\mathbb{C}^3$ coordinates are given by

$$z = r\mathrm{e}^{\mathrm{i}\theta}\,, \quad (x, y) = \sqrt{1-r^2}(x_0, y_0)\,. \tag{F.3}$$

The target space metric ($G_{\mu\nu}$ in (2.8)) takes the form

$$\begin{aligned} \mathrm{d}s^2 &= -\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-r^2} + r^2\mathrm{d}\phi^2 + (1-r^2)(\mathrm{d}S^3)^2 \\ &= -\mathrm{d}t^2 + \frac{\bar{z}^2\mathrm{d}z^2 + 2(2-z\bar{z})\mathrm{d}z\mathrm{d}\bar{z} + z^2\mathrm{d}\bar{z}^2}{4(1-z\bar{z})} + (1-z\bar{z})(|\mathrm{d}x_0|^2 + |\mathrm{d}y_0|^2)\,. \end{aligned} \tag{F.4}$$

We choose t, σ^i as world volume coordinates on the brane (σ^α in (2.8)). Curves in the neighbourhood of (F.1) may be characterized by specifying the function $z = z(\sigma^i, t)$ (intuitively, z parameterizes the two transverse fluctuations of the brane away from its ambient S^3 shape). The fluctuation $z(\sigma^i, t)$ corresponding to linear polynomials (5.1) is given, to first order in $\delta a, \delta b$ and δc by

$$\begin{aligned} z &= \frac{\mathrm{e}^{\mathrm{i}t}}{c_0} - \left(\frac{\mathrm{e}^{\mathrm{i}t}}{c_0^2}\delta c + \frac{\sqrt{1-z\bar{z}}}{c_0}(x_0\delta a + y_0\delta b)\right) + \mathcal{O}(2)\,, \\ \dot{z} &= \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}t}}{c_0} + \mathcal{O}(1)\,, \\ \partial_i z &= \mathcal{O}(1)\,. \end{aligned} \tag{F.5}$$

equation (5.1) together with (2.8) and (F.4) may then be used to determine the induced metric on the world volume of the brane; at $t = 0$ and to zeroth order in fluctuations we find:

$$\begin{aligned} g_{ij} &= (1-r^2)g^{S^3}_{ij} + \mathcal{O}(2)\,, \\ g_{0i} &= \mathcal{O}(1)\,, \\ g_{00} &= -(1-r^2) + \mathcal{O}(1)\,. \end{aligned} \tag{F.6}$$

F.2 ω_{BI}

Let $\omega_{\mathrm{BI}} = \mathrm{d}\theta_{\mathrm{BI}}$ denote the contribution of the Born-Infeld term in the action to the symplectic form. We have

$$\begin{aligned} \theta_{\mathrm{BI}} &= \frac{N}{2\pi^2}\int \mathrm{d}^3\sigma\,(p_z\,\delta z + p_{\bar{z}}\,\delta\bar{z})\,, \\ \text{where}\quad p_z &= \sqrt{-g}g^{00}(\dot{z}G_{zz} + \dot{\bar{z}}G_{z\bar{z}}) \\ &= -\left(\dot{z}\frac{\bar{z}^2}{4} + \dot{\bar{z}}\frac{2-z\bar{z}}{4}\right). \end{aligned} \tag{F.7}$$

We have dropped factors of $\sqrt{g^{S^3}}$, absorbing them into the integration measure $\mathrm{d}^3\sigma$.

Using (F.5) and the integrals:

$$\int \mathrm{d}^3\sigma\,1 = 2\pi^2\,, \qquad \int \mathrm{d}^3\sigma\,|x_0|^2 = \int \mathrm{d}^3\sigma\,|y_0|^2 = \pi^2\,, \tag{F.8}$$

with all other linear and quadratic integrals evaluating to zero, we find

$$\theta_{\rm BI} = N\left(\frac{1}{|c_0|^4} - \frac{1}{|c_0|^6}\right)\frac{\bar{c}_0\,\delta c - c_0\,\delta\bar{c}}{2{\rm i}}\,.$$

We can find $\theta_{\rm BI}$ at a general point with the replacement

$$\bar{c}_0\,\delta c \to \bar{c}^i\,{\rm d}c_i\,, \quad c_0\,\delta\bar{c} \to c_i\,{\rm d}\bar{c}^i\,, \quad |c_0|^2 \to \bar{c}^i c_i \equiv |c|^2\,, \tag{F.9}$$

yielding

$$\theta_{\rm BI} = N\left(\frac{1}{|c|^4} - \frac{1}{|c|^6}\right)\frac{\bar{c}^i\,{\rm d}c_i - c_i\,{\rm d}\bar{c}^i}{2{\rm i}}\,. \tag{F.10}$$

Taking the exterior derivative gives

$$\omega_{\rm BI} = 2N\left[\left(\frac{1}{|c|^4} - \frac{1}{|c|^6}\right)\frac{{\rm d}\bar{c}^i\wedge{\rm d}c_i}{2{\rm i}} + \left(\frac{3}{|c|^6} - \frac{2}{|c|^4}\right)\frac{\bar{c}^i c_j}{|c|^2}\frac{{\rm d}\bar{c}^j\wedge{\rm d}c_i}{2{\rm i}}\right]. \tag{F.11}$$

Note that

$$\omega_{\rm BI} = f_{\rm BI}(|c|^2)\frac{{\rm d}\bar{c}^i\wedge{\rm d}c_i}{2{\rm i}} + f'_{\rm BI}(|c|^2)\bar{c}^i c_j\frac{{\rm d}\bar{c}^j\wedge{\rm d}c_i}{2{\rm i}}\,; \qquad f_{\rm BI}(|c|^2) = 2N\left(\frac{1}{|c|^4} - \frac{1}{|c|^6}\right). \tag{F.12}$$

F.3 $\omega_{\rm WZ}$

Using the metric (F.4), the volume of the giant graviton is $2\pi^2(1-r^2)^{3/2}$. To find the volume swept out when it is deformed, we need only consider the deformations perpendicular to its surface. About our special point in parameter space, this gives:

$$\omega_{\rm WZ} = \frac{2N}{\pi^2}\int{\rm d}^3\sigma\,(1-r^2)r\delta r\wedge\delta\theta = \frac{2N}{\pi^2}\int{\rm d}^3\sigma\,(1-z\bar{z})\frac{\delta\bar{z}\wedge\delta z}{2{\rm i}}\,.$$

If we use (F.5) and (F.8), this becomes:

$$\omega_{\rm WZ} = \frac{4N}{|c_0|^4}\left(1 - \frac{1}{|c_0|^2}\right)\frac{\delta\bar{c}\wedge\delta c}{2{\rm i}} + \frac{2N}{|c_0|^2}\left(1 - \frac{1}{|c_0|^2}\right)^2\left(\frac{\delta\bar{a}\wedge\delta a}{2{\rm i}} + \frac{\delta\bar{b}\wedge\delta b}{2{\rm i}}\right).$$

Making the replacement

$$\delta\bar{c}\wedge\delta c \to \frac{\bar{c}^i c_j}{|c|^2}{\rm d}\bar{c}^j\wedge{\rm d}c_i\,, \qquad \delta\bar{a}\wedge\delta a + \delta\bar{b}\wedge\delta b \to {\rm d}\bar{c}^i\wedge{\rm d}c_i - \frac{\bar{c}^i c_j}{|c|^2}{\rm d}\bar{c}^j\wedge{\rm d}c_i\,, \tag{F.13}$$

we get the Wess-Zumino contribution to the symplectic form at an arbitrary point:

$$\omega_{\rm WZ} = 2N\left[\left(\frac{1}{|c|^2} - \frac{2}{|c|^4} + \frac{1}{|c|^6}\right)\frac{{\rm d}\bar{c}^i\wedge{\rm d}c_i}{2{\rm i}} - \left(\frac{1}{|c|^2} - \frac{4}{|c|^4} + \frac{3}{|c|^6}\right)\frac{\bar{c}^i c_j}{|c|^2}\frac{{\rm d}\bar{c}^j\wedge{\rm d}c_i}{2{\rm i}}\right]. \tag{F.14}$$

The analogue of (F.12) also applies to this case upon defining

$$f_{\rm WZ} = 2N\left(\frac{1}{|c|^2} - \frac{2}{|c|^4} + \frac{1}{|c|^6}\right). \tag{F.15}$$

F.4 $\omega_{\rm full}$

Adding together (F.11) and (F.14) gives

$$\omega_{\rm full} = 2N\left[\left(\frac{1}{|c|^2}-\frac{1}{|c|^4}\right)\frac{{\rm d}\bar{c}^i\wedge{\rm d}c_i}{2{\rm i}} - \left(\frac{1}{|c|^2}-\frac{2}{|c|^4}\right)\frac{\bar{c}^i c_j}{|c|^2}\frac{{\rm d}\bar{c}^j\wedge{\rm d}c_i}{2{\rm i}}\right]. \tag{F.16}$$

Defining

$$f_{\rm full} = 2N\left(\frac{1}{|c|^2}-\frac{1}{|c|^4}\right), \tag{F.17}$$

the analogue of (F.12) applies.

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